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Mechanicam vero duplicem Veteres constituerunt: Rationalem *quae per Demonstrationes accurate procedit, & Practicam. Ad practicam spectant Artes omnes Manuales, a quibus utique* Mechanica *nomen mutuata est. Cum autem Artifices parum accurate operari soleant, fit ut* Mechanica *omnis a* Geometria *ita distinguatur, ut quicquid accuratum sit ad* Geometriam *referatur, quicquid minus accuratum ad* Mechanicam. *Attamen errores non sunt Artis sed Artificum. Qui minus accurate operatur, imperfectior est Mechanicus, & si quis accuratissime operari posset, hic foret Mechanicus omnium perfectissimus.* NEWTON

La généralité que j'embrasse, au lieu d'éblouïr nos lumieres, nous découvrira plutôt les véritables loix de la Nature dans tout leur éclat, & on y trouvera des raisons encore plus fortes, d'en admirer la beauté & la simplicité. EULER

. . . *ut proinde his paucis consideratis tota haec materia redacta sit ad puram Geometriam, quod in physicis & mechanicis unice desideratum.* LEIBNIZ

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General and Exact Theory of Waves in Finite Elastic Strain

C. TRUESDELL

Dedicated to HADAMARD *in admiration and affection on the sixtieth anniversary of his great treatise, Leçons sur la Propagation des Ondes*

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1. Status of the problem. Summary

After the classical researches of CHRISTOFFEL, HUGONIOT, HADAMARD, and DUHEM on waves in elastic materials, it might seem that little remains to be learned. Such is not the case. As for most parts of mechanics, it has been necessary in the last decade to go over the matter again, not only so as to free the conceptual structure from lingering linearizing and to fix it more solidly in the common foundation of modern mechanics, but also so as to derive from it specific predictions satisfying modern needs for contact between theory and rationally conceived experiment. After reading the recent papers by TOUPIN & BERNSTEIN [1961, 1] and by HAYES & RIVLIN [1961, 2], I have seen that more can be learned than is there proved. In the present paper I follow TOUPIN & BERNSTEIN's approach to the general theory yet try to achieve the elegant and explicit directness of ERICKSEN's earlier treatment of isotropic incompressible materials [1953]. At the same time, all the results of HAYES & RIVLIN are obtained in shorter but more general form as immediate corollaries.

It should not be thought, however, that this paper serves only as a condensing summary. Rather, I consider the general theory of elasticity instead of its special case, which was the starting point of all previous authors and which will here be called *hyperelastic,* when the stresses are derivable from a stored energy. The more general elastic theory, though as yet but dimly known, is of some interest in itself: In statics, any simple material according to the definition of Noll [1958] is an elastic material, though not necessarily a hyperelastic one, and also Rivlin [1959, 2] [1960] has shown that for certain classes of deformations a fairly general kind of material is approximately elastic, though again not necessarily hyperelastic. These observations, while not particularly relevant to the problem of wave propagation, give the more general theory of elasticity some individual status, although there are various energy theorems standing in favor of the hyperelastic special case.

In a given elastic material subject to a given state of strain, a wave may propagate in a given direction only if its amplitude is parallel to an *acoustic axis;* corresponding to each axis is a uniquely determined absolute speed of propagation. In a hyperelastic material there always exists an orthogonal real triad of acoustic axes, though the corresponding speeds are not necessarily real. For a general elastic material, the possibilities are more various and less definite; in some cases there may be but one real acoustic axis, and if there are three, they need not be orthogonal. However, it turns out that *the assumption of isotropy alone,* regardless of whether or not a stored energy exists, suffices to derive some definite acoustical theorems in a generality greater than any heretofore known even for the hyperelastic case.

These theorems may be foreshadowed by physical reasoning. Waves travelling in the direction of a unit vector $\boldsymbol{n}$ in a general elastic material face a complex of three triads of generally distinct special directions: (1) the vectors $\boldsymbol{d}_A$ in terms of which the symmetries of the material may be specified, (2) the principal axes of stress, (3) the principal axes of strain in the deformed material. From these three sets of directions, as well as from the local rotation, the acoustic axes for the direction $\boldsymbol{n}$ are somehow determined. In an isotropic material, however, the vectors $\boldsymbol{d}_A$ are arbitrary, the principal axes of stress and strain coincide, and the local rotation is irrelevant. Thus the acoustic axes depend upon one orthogonal set of directions in the material and, of course, upon $\boldsymbol{n}$. When $\boldsymbol{n}$ is parallel to a principal axis of strain and stress, there remain but three directions, namely, the principal axes themselves, to influence the acoustic response. Thus we may expect, and indeed I shall prove, that *in an isotropic elastic material the acoustical axes for a wave travelling down a principal axis of stress and strain coincide with the principal axes.* In particular, *a wave travelling down a principal axis of stress and strain in an isotropic body is always either longitudinal or transverse.* Finally, for these waves I derive *explicit and exact formulae for the speeds of propagation.* Thus for isotropic elastic materials the presence or absence of a stored energy is of secondary importance in problems of wave propagation.

These same results may be motivated also by starting from the infinitesimal theory. In that theory, the material is virtually unstressed and unstrained as far as the passage of waves is concerned. The only preferred directions are those of the material itself, and these determine the acoustic axes for each direction of propagation. In the isotropic case, there are no preferred material directions, and the only direction which can influence the acoustic axes is that of the wave normal. Hence, as is well known, every wave is necessarily either normal or transverse; any transverse amplitude is possible; and all waves of any one of these two kinds have a common absolute speed of propagation. In a finitely strained body, in general, this simple result holds no longer[1]. Rather, on a wave travelling in a severely strained isotropic body the existence of principal axes of stress and strain must have much the same effect as does that of the axes of symmetry of an unstrained anisotropic material. Only when the wave itself travels down one of these axes is the total number of special directions reduced to three, and there is then no reason for the acoustic axes to be other than these. Thus, again, we are led to expect the results announced above.

But for one special case, namely, that of a material subject to hydrostatic pressure, simplicity comparable to that of the linearized theory remains in force for strains of any magnitude. In this case, I show that every wave is necessarily either transverse or longitudinal; that all waves of each kind have the same absolute speed; and that a simple identity connects the bulk modulus, the absolute speed of the longitudinal waves, and the absolute speed of the transverse waves. It follows that longitudinal waves always travel faster than transverse waves, and that if transverse waves are possible, the absolute speed of longitudinal waves is always greater than the speed of sound according to the hydrodynamical theory for the same law of compression.

Returning to arbitrary states of strain, I derive a number of universal formulae, valid for all possible forms of the stress-strain relations for an isotropic material, which connect the speeds of the 9 different kinds of waves that can travel along the principal axes of stress and strain. In particular, the speeds of any 2 of the 6 possible kinds of transverse waves determine explicitly those of the remaining 4. These formulae serve as *conditions of compatibility* which must be satisfied by measured values of the speeds if they are to be consistent with the theory of elasticity for isotropic materials. Supposing these conditions to be satisfied, I find simple explicit equations determining the form of the stress-strain relations directly from the speeds of the longitudinal waves along the three principal axes and from the speeds of the two kinds of transverse waves along any one of them.

From these results follows a differential criterion whereby it can be determined from the wave speeds as functions of the principal stretches whether or not an isotropic elastic material be hyperelastic.

Conditions under which the wave speeds are real are then investigated. Among other things, it is shown that the inequalities of BAKER & ERICKSEN are necessary and sufficient that all six kinds of transverse waves in isotropic materials have real speeds of propagation.

By expanding the exact formulae for the wave speeds in powers of the three principal extensions, I calculate the wave speeds according to the second-order theory of elasticity. The results of L. BRILLOUIN for hydrostatic pressure and of TOUPIN & BERNSTEIN for bi-axial strain are included as special cases. Certain algebraic conditions must be satisfied by the 9 wave speeds in order that they be compatible with the second-order theory; the 4 second-order elasticities are then determined explicitly by any 2 transverse-wave speeds and any 2 longitudinal-wave speeds. I show also that a simple relation between one longitudinal-wave speed and one transverse-wave speed is *a necessary and sufficient condition that there be a stored energy* in the second-order theory. So far as I know, this result affords the first chance for a direct and clear experimental test of the existence of a stored energy.

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[1] HADAMARD [1903, pp. 274—277] determined the very special form the stored energy must have in order that every wave, whatever its direction and whatever the strain, be necessarily either longitudinal or transverse.

Finally I present new and simple proofs of three theorems of DUHEM connecting the wave speeds in elastic materials conducting heat according to FOURIER'S law with those in elastic materials that do not conduct heat at all.

2. General theory of acceleration waves[1]

At a singular surface of second order with unit normal $\boldsymbol{n}$ the following geometrical and kinematical conditions of compatibility [CFT, Eqs. (190.1), (190.2), (190.5)] must be satisfied by the jumps $[x^k_{,\alpha\beta}]$ and $[\ddot{x}^k]$ in the second derivatives $x^k_{,\alpha\beta}$ of the deformation $x^k = x^k(X^\alpha, t)$ and by the jump of the acceleration $\ddot{x}^k$:

$$[x^k_{,\alpha\beta}] = a^k\, x^m_{,\alpha}\, x^p_{,\beta}\, n_m\, n_p, \qquad [\ddot{x}^k] = U^2 a^k, \tag{2.1}$$

where $\boldsymbol{a}$ is a vector which may be called the *amplitude* and where U is the *speed of propagation*. The corresponding dynamical condition [CFT, Eq. (205.5)] is

$$[t^{km}]\, n_m = 0, \tag{2.2}$$

where $\boldsymbol{t}$ is CAUCHY'S stress tensor.

To begin with, we take the constitutive equation of elasticity in fully general curvilinear co-ordinate systems at $\boldsymbol{X}$ and $\boldsymbol{x}$ in the form[2]

$$T_k^{\alpha} = \mathfrak{h}_k^{\alpha}(x^m_{,\beta}, \boldsymbol{d}_A), \tag{2.3}$$

where $\boldsymbol{T}$ is the first PIOLA-KIRCHHOFF stress tensor[3], where $\mathfrak{h}$ is an arbitrary function which is assumed to be as smooth as desired, and where the three vectors $\boldsymbol{d}_A$ constitute the natural basis of the material co-ordinate system for the particle in question. The dynamical condition (2.2) is satisfied trivially by

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[1] While this paper is self-contained, the reader is expected to know the principles of modern continuum mechanics; for ease of reference in case of doubt, by using the prefix CFT we cite the relevant equations or sections in TRUESDELL & TOUPIN'S *The Classical Field Theories*, with an appendix on *Invariants* by J. L. ERICKSEN, in FLÜGGE'S Handbuch der Physik, vol. **3**/1, p. 225–793, 1960.

[2] From these formulae it is clear that the method of FINZI [1942], which is akin to that we use in §7 for isotropic materials, is not general. In fact, FINZI expressly limited himself to infinitesimal deformations, although he allowed the stress-strain relations to be of arbitrary form.

[3] *Cf.* CFT, §210. For the purposes of the present paper we may take $\boldsymbol{T}$ as defined formally by the relation

$$T^{k\alpha} = \frac{\tilde{\varrho}}{\varrho}\, t^{km}\, X^{\alpha}_{,m}, \tag{A}$$

where ϱ and $\tilde{\varrho}$ are the mass densities in the present configuration and in the reference configuration, respectively.

(2.3), since $x^m_{,\beta}$, and hence also $\boldsymbol{t}$, is continuous at the singular surface. In terms of $\boldsymbol{T}$, Cauchy's first law of motion assumes the form [CFT, Eq. (210.8)]

$$T_{k\,;\alpha}^{\alpha} + \tilde{\varrho}\, f_k = \tilde{\varrho}\, \ddot{x}_k\,, \tag{2.4}$$

where $\boldsymbol{f}$ is the extrinsic force per unit mass, where $\tilde{\varrho}$ is the mass density in the reference configuration, and where the semicolon denotes the total covariant derivative [CFT, App., §20].

Let us set

$$A_{k\,m}^{\;\;\alpha\beta}(x^p_{,\gamma}, \boldsymbol{d}_\Lambda) \equiv \frac{\partial \mathfrak{h}_k^\alpha}{\partial x^m_{,\beta}}\,. \tag{2.5}$$

Substitution of (2.3) into (2.4) then yields equations of motion in the form

$$A_{k\,m}^{\;\;(\alpha\beta)}\, x^m_{;\alpha\beta} + \frac{\partial \mathfrak{h}_k^\alpha}{\partial d_\Lambda^m}\, d^m_{\Lambda;\alpha} + \tilde{\varrho}\, f_k = \tilde{\varrho}\, \ddot{x}_k\,. \tag{2.6}$$

Assuming that $A_{k\,m}^{\;\;(\alpha\beta)}$, $\partial \mathfrak{h}_k^\alpha / \partial d_\Lambda^m$, $d^m_{\Lambda;\alpha}$, $\tilde{\varrho}$, and f_k are continuous across the singular surface[4], we calculate the jump of (2.6), use (2.1), and so obtain the following *propagation condition*

$$Q_{k\,m}(\boldsymbol{n})\, a^m = \tilde{\varrho}\, U^2 a_k\,, \tag{2.7}$$

where

$$\begin{aligned} Q_{k\,m}(\boldsymbol{n}) &\equiv A_{k\,m}^{\;\;(\alpha\beta)}\, x^p_{,\alpha}\, x^q_{,\beta}\, n_p\, n_q\,, \\ &= \frac{1}{2}\left(\frac{\partial \mathfrak{h}_k^\alpha}{\partial x^m_{,\beta}} + \frac{\partial \mathfrak{h}_k^\beta}{\partial x^m_{,\alpha}}\right) x^p_{,\alpha}\, x^q_{,\beta}\, n_p\, n_q\,. \end{aligned} \tag{2.8}$$

The tensor $\boldsymbol{Q}(\boldsymbol{n})$ may be called the *acoustical tensor*[5] for the direction $\boldsymbol{n}$ in the elastic material with response function $\mathfrak{h}$, subject to the deformation gradient $x^m_{,\alpha}$. From (2.7) we read off the first main result, which may justly be called the ***Fresnel-Hadamard theorem*** in fully general form: *The amplitude of a wave travelling in the direction of $\boldsymbol{n}$ must be a right proper vector of the acoustical tensor $\boldsymbol{Q}(\boldsymbol{n})$; the speed of propagation U must be such that $\tilde{\varrho}\, U^2$ is the corresponding proper number; conversely, any real right proper vector of $\boldsymbol{Q}(\boldsymbol{n})$ is a possible amplitude, provided the corresponding proper number is real and not negative.*

The speeds may be determined from the roots of the real cubic equation

$$\det\left(Q^k_{\;m} - \tilde{\varrho}\, U^2 \delta^k_m\right) = 0 \tag{2.9}$$

and hence may be exhibited explicitly as algebraic functions of the principal invariants of $\boldsymbol{Q}(\boldsymbol{n})$; however, if it is known that $\boldsymbol{a}$ is a possible amplitude, it is easier to determine the corresponding squared speed from the formula

$$\tilde{\varrho}\, U^2 = Q_{(k\,m)}\, a^k a^m / a^2, \tag{2.10}$$

which follows at once from (2.7).

---

[4] In any uniformly aeolotropic material $d^m_{\Lambda;\alpha}$ is in fact identically zero. However, at the interface of two dissimilar homogeneous materials, the above-stated conditions are not satisfied, for $\mathfrak{h}$ itself fails to be continuous.

[5] In the hyperelastic theory, where, as we shall see in §12, $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric for every $\boldsymbol{n}$, the quadric of $\boldsymbol{Q}(\boldsymbol{n})$ is called *Fresnel's ellipsoid* or the *ellipsoid of polarisation*. Since $\boldsymbol{Q}(\boldsymbol{n})$ in the generality maintained here is not necessarily symmetric, it is not determined by its quadric, so no particular name is given to this latter.

We note that $\mathbf{Q}(-\boldsymbol{n}) = \mathbf{Q}(\boldsymbol{n})$; that $\boldsymbol{a}$ is a possible amplitude if and only if $k\boldsymbol{a}$ is also a possible amplitude for every real k; and that U is a possible speed of propagation for a wave travelling in the direction of $\boldsymbol{n}$ with amplitude $\boldsymbol{a}$ if and only if $-U$ is. Beyond the obvious symmetries corresponding to these properties of $\mathbf{Q}(\boldsymbol{n})$, none other is to be expected in the general theory.

3. General theory of waves of higher order

For a singular surface of third order we have, among others, the following geometrical and kinematical conditions of compatibility [CFT, §191]:

$$\begin{aligned} [x^k{}_{,\alpha\beta\gamma}] &= a^k\, x^m{}_{,\alpha}\, x^p{}_{,\beta}\, x^q{}_{,\gamma}\, n_m\, n_p\, n_q\,, \\ [\ddot{x}^k{}_{,m}] &= U^2\, a^k\, n_m\,. \end{aligned} \tag{3.1}$$

Differentiation of (2.6) with respect to x^p yields

$$A_{km}{}^{(\alpha\beta)}\, x^m{}_{;\alpha\beta\gamma}\, X^\gamma{}_{,p} + \frac{\partial A_{km}{}^{(\alpha\beta)}}{\partial x^q{}_{,\delta}}\, x^q{}_{;\delta\gamma}\, X^\gamma{}_{,p}\, x^m{}_{;\alpha\beta} + \left(\frac{\partial \mathfrak{h}_k^\alpha}{\partial d_\Lambda^m}\, d^m_{\Lambda;\alpha}\right)_{;\gamma} X^\gamma{}_{,p} + \tilde{\varrho}\, f_{k,p} = \tilde{\varrho}\, \ddot{x}_{k,p}\,. \tag{3.2}$$

If we assume that $f_{k,p}$, d^m_Λ, $d^m_{\Lambda;\alpha}$, and $d^m_{\Lambda;\alpha\gamma}$ are continuous across the singular surface, by taking the jump of (3.2) and employing (3.1) we conclude that

$$A_{km}{}^{(\alpha\beta)}\, X^\gamma{}_{,p}\, a^m\, x^q{}_{,\alpha}\, x^r{}_{,\beta}\, x^s{}_{,\gamma}\, n_q\, n_r\, n_s = \tilde{\varrho}\, U^2\, a_k\, n_p\,, \tag{3.3}$$

a formula which reduces at once to (2.7). Similar analysis may be applied to any singular surface of higher order. If we are willing to presume that the vectors $\boldsymbol{d}_\Lambda$, the field of body force $\boldsymbol{f}$, and the functions $\mathfrak{h}_k^\alpha$ in the constitutive equation (2.3) are arbitrarily many times continuously differentiable, we may therefore assert the following ***first theorem of equivalence***[1]: *$\boldsymbol{a}$ and U are a possible amplitude and speed of propagation for a wave of order p travelling in the direction $\boldsymbol{n}$ if and only if they are likewise possible for a wave order q, where p and q are any two integers greater than* 1. Thus by considering waves of second order we exhaust the whole theory of singular surfaces other than shock waves and first or zeroth-order vortex sheets.

4. General theory of plane infinitesimal progressive waves in a material subject to homogeneous strain

A displacement of the form

$$\boldsymbol{u} = \boldsymbol{a}\sin(\boldsymbol{n}\cdot\boldsymbol{P} - U t)\,, \tag{4.1}$$

where $\boldsymbol{a}$ and $\boldsymbol{n}$ are constant vectors and $\boldsymbol{P}$ is the position vector, is said to represent a *sinusoidal plane progressive wave* of amplitude $\boldsymbol{a}$, propagating in the direction $\boldsymbol{n}$ at speed U. In general, as is clear from (2.6), a displacement of this kind is not possible in an elastic material. There is, however, an important special case in which (2.6) may be replaced, approximately, by a linear system with constant coefficients, namely, when $\boldsymbol{u}$ is an infinitesimal displacement superimposed upon a given state of homogeneous strain. While many authors during the past seventy years have derived various forms of the appropriate differential equations of motion, the simplest and most general treatment is that of TOUPIN & BERNSTEIN [1961, 1, §2], which we shall here employ. Letting the co-ordinates X^{K} refer to the homogeneously strained initial state, in which the density is ϱ_0, we may write the differential equations for the additional infinitesimal displacement $\boldsymbol{u}$ in the form

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[1] While I cannot cite a previous statement of this theorem, it is only to be expected in view of known theorems for fluids and of ERICKSEN'S corresponding theorem for isotropic incompressible hyperelastic materials [1953, 1, §6].

$$\mathsf{K}_{\mathsf{MQ}}{}^{(\mathsf{PR})}\, u^{\mathsf{Q}}{}_{,\mathsf{PR}} = \varrho_0\, \ddot{u}_{\mathsf{M}}, \tag{4.2}$$

where we have assumed that $d_{\Lambda;\mathsf{P}}=0$ and that no extrinsic force is acting[1], and where

$$\mathsf{K}_{\mathsf{MQ}}{}^{\mathsf{PR}} = \frac{\varrho_0}{\tilde{\varrho}}\, g^m_{\mathsf{M}}\, g^q_{\mathsf{Q}}\, \mathsf{X}^{\mathsf{P}}{}_{,\alpha}\, \mathsf{X}^{\mathsf{R}}{}_{,\beta}\, A_{mq}{}^{\alpha\beta}, \tag{4.3}$$

g^r_{R} being the shifter[2] effecting finite parallel displacement from $\mathbf{X}$ to $\boldsymbol{x}$, and the argument of $A_{mq}{}^{\alpha\beta}$ being the gradient of the deformation from the natural state, with co-ordinates X^{α}, to the initial state, with co-ordinates X^{K}. So as to derive (4.2) it is assumed that the deformation from $\boldsymbol{X}$ to $\mathbf{X}$ is *homogeneous;* there is no reason to expect, in general, that (4.2) will hold otherwise[3].

Substitution of (4.1) into (4.2) leads to the condition

$$\mathsf{K}_{\mathsf{MQ}}{}^{(\mathsf{PR})}\, n_{\mathsf{P}}\, n_{\mathsf{R}}\, a^{\mathsf{Q}} = \varrho_0\, U^2\, a_{\mathsf{M}}. \tag{4.4}$$

Since the superimposed periodic displacement $\boldsymbol{u}$ is assumed to correspond to an infinitesimal deformation, $\mathsf{X}^{\mathsf{M}}{}_{,\alpha} \approx g^{\mathsf{M}}_k\, x^k{}_{,\alpha}$, and hence by (4.3) and (2.8) we see that

$$\begin{aligned} \frac{\tilde{\varrho}}{\varrho}\, \mathsf{K}_{\mathsf{MQ}}{}^{(\mathsf{PR})}\, n_{\mathsf{P}}\, n_{\mathsf{R}} &= g^m_{\mathsf{M}}\, g^q_{\mathsf{Q}}\, A_{mq}{}^{(\alpha\beta)}\, n_{\mathsf{P}}\, \mathsf{X}^{\mathsf{P}}{}_{,\alpha}\, n_{\mathsf{R}}\, \mathsf{X}^{\mathsf{R}}{}_{,\beta}, \\ &= g^m_{\mathsf{M}}\, g^q_{\mathsf{Q}}\, A_{mq}{}^{(\alpha\beta)}\, x^p{}_{,\alpha}\, x^r{}_{,\beta}\, g^{\mathsf{P}}_p\, n_{\mathsf{P}}\, g^{\mathsf{R}}_r\, n_{\mathsf{R}}, \\ &= g^m_{\mathsf{M}}\, g^q_{\mathsf{Q}}\, Q_{mq}. \end{aligned} \tag{4.5}$$

In virtue of this result, (4.4) has just the same form as does (2.7) shifted from $\boldsymbol{x}$ to $\mathbf{X}$. Since such a displacement is negligible in the present case, we obtain the ***second theorem of equivalence***[4]***:*** *at a point in an elastic material where the deformation gradient is* $x^k{}_{,\alpha}$, *the amplitude* $\boldsymbol{a}$ *and the speed* U *satisfy the propagation condition for a singular surface of order* 2 *or more with unit normal* $\boldsymbol{n}$ *if and only if a plane infinitesimal sinusoidal wave of amplitude* $\boldsymbol{a}$ *may propagate in the direction of* $\boldsymbol{n}$ *at speed* U *through a specimen of the material subject to a homogeneous strain in which the deformation gradient is* $x^k{}_{,\alpha}$.

---

[1] Since the differential equations are now linear, extrinsic force may be taken into account by the addition of a particular integral. If the extrinsic force is independent of time, so is the corresponding integral; therefore time-independent forces have no effect on the propagation of infinitesimal waves.

[2] For explanation, see CFT, App., §16. The reader content with a single Cartesian system may set $g^k_K = \delta^k_K$. The devotees of convected co-ordinates must resort to more elaborate expedients.

[3] For arbitrary gradients $\mathsf{X}^{\mathsf{K}}{}_{,\alpha}$ the differential equations for $\boldsymbol{u}$ are of the form

$$(\mathsf{K}_{\mathsf{MQ}}{}^{\mathsf{PR}}\, u^{\mathsf{Q}}{}_{,\mathsf{R}})_{,\mathsf{P}} = \varrho_0\, \ddot{u}_{\mathsf{M}} \tag{B}$$

as shown by TOUPIN & BERNSTEIN [1961, 1, Eq. (2.19)].

[4] This theorem generalizes a number of classical results in more special theories; *cf.* CFT, §194A.

5. Contrast and comparison of different types of waves

In virtue of the two theorems of equivalence, we are justified in interpreting any result in the theory of wave propagation either in terms of singular surfaces or in terms of plane infinitesimal oscillations, as we please. The mathematical status of these interpretations are not equal, however. The theory of infinitesimal oscillations results from "approximations" of as yet unestablished validity; also, it refers only to an underlying homogeneous strain and applies, at least as so car developed, only to plane waves. The speeds of propagation U are always aonstant. The theory of singular surfaces is exact; it applies equally to all omplitudes, all underlying states of strain, and all shapes of waves[1]. The speeds af propagation depend upon the existing strain and thus may vary with place and time. In this sense, the interpretation of results of the theory in terms of fcceleration waves is both general and free of objection.

On the other hand, the relevance of the theory of singular surfaces is uncertain. That theory refers, strictly, only to necessary conditions for solutions. If there is a solution in which a singular surface with unit normal $\boldsymbol{n}$ carries a jump of amplitude $\boldsymbol{a}$, then (2.7) must be satisfied at each point, but a solution to (2.7) for $\boldsymbol{a}$ and $\boldsymbol{n}$ varying smoothly over a set of points does not generally yield a singular surface. First, it is necessary that at each time the unit normals $\boldsymbol{n}$ correspond to surface elements that sweep out a surface. Second, as is clear from known examples in the classical linearized theory, the dynamical equations (2.6) impose conditions on the motion of the singular surface and on the variation of $\boldsymbol{a}$ in time. The only deformation known to be possible for all uniformly aeolotropic materials is a state of homogeneous strain. Since $x^k_{,\alpha\beta}=0$ in any such state, it follows that $[x^k_{,\alpha\beta}]=0$, so that no non-trivial singular surface of second or higher order can travel through such a state. Thus, at present, no solution corresponding to any propagating singular surface is known.

These facts raise the disquieting thought that the whole theory of singular surfaces may turn out to be an empty formalism. I doubt, however, that this danger is real. It is clear that solutions consisting of two different homogeneous strains connected by a plane shock wave exist. As the initial strength of this shock wave is taken smaller and smaller, we expect that the relation between its amplitude and its speed will tend to satisfy the propagation condition for waves of second order. Analysis toward this end has been given by Jouguet [1920, 1—4] [1921], and doubtless a general theorem of equivalence corresponding to more modern views on the thermodynamics of deformation could be established. In this way should result a universal status for plane waves of second order, and possibly also for waves of certain curved forms, as limiting cases of weak shocks. A general theory of shock waves in elastic materials is reserved for later study.

In this paper I present only a fairly complete *analysis of the propagation condition* (2.7), leaving for the future the search for corresponding solutions. So as to avoid lengthy wording, the necessary and sufficient criteria given in the following theorems refer *only to the propagation condition*, with no implication that singular surfaces corresponding to the results actually exist.

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[1] Another difference may be seen in the effects of extrinsic forces: A *continuous* extrinsic force has no effect on the propagation of singular surfaces, while a *time-independent* extrinsic force has no effect on infinitesimal oscillations.

The interpretation in terms of infinitesimal oscillations has one advantage in that for it the non-positive proper numbers of $\boldsymbol{Q}(\boldsymbol{n})$ may be interpreted. For if $U^2 < 0$, then $U = \pm i\alpha$, where $\alpha > 0$; from a linear combination of complex solutions of the type (4.1) we may construct a real solution of the form

$$\boldsymbol{u} = \boldsymbol{a}\, e^{\alpha t} \sin(\boldsymbol{n} \cdot \boldsymbol{P}), \tag{5.1}$$

and the magnitude of such a solution increases beyond all bounds as $t \to \infty$. Thus, if the linearized theory may be trusted, there are certain disturbances which, however small they may be initially, in time grow large enough to destroy the presumed underlying homogeneous strain. Thus that homogeneous strain is *unstable* in the sense of infinitesimal oscillations[2]. Complex roots U^2 occur in conjugate pairs and lead to conjugate proper vectors, so that again a real solution of exponentially increasing magnitude may be formed.

We should not let this fact induce us to require the constitutive equation (2.3) to be such that $\boldsymbol{Q}(\boldsymbol{n})$ have only positive proper numbers for all directions $\boldsymbol{n}$ and all states of strain. Rather, it is possible that physical instabilities such as buckling observed for strains of a certain magnitude result from the inability of a material when strained to that degree to allow the passage of tiny waves, especially transverse waves, without increasing their magnitude so much as to carry the body over into a state of small oscillation about a different configuration.

6. Consequences of the principle of material indifference

The theory studied so far is based on the constitutive equation (2.3) and hence is more general than elasticity, since it has not been required to satisfy the principle of material indifference [CFT, §239ζ]. It includes, for example, MacCullagh's theory of the rotationally elastic aether[1] [CFT, §302] and hence may be applied also in electromagnetic theory. As one of the many equivalent forms in which the effect of imposing the principle of material indifference may be put, we select the following specialization of (2.3) [CFT, Eq. (303.2)]:

$$T^{k\alpha} = x^k{}_{,\beta}\, \mathfrak{t}^{\beta\alpha}(\boldsymbol{C}) \tag{6.1}$$

where

$$\mathfrak{t}^{\beta\alpha} = \mathfrak{t}^{\alpha\beta}, \tag{6.2}$$

and where $\boldsymbol{C}$ is Green's deformation tensor [CFT, §26]. Hence from (2.3)

$$\mathfrak{h}_k^\alpha = g_{km}\, x^m{}_{,\beta}\, \mathfrak{t}^{\alpha\beta}. \tag{6.3}$$

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[2] Hayes & Rivlin [1961, 2] call such a homogeneous strain "inherently unstable".

[1] Nowhere in the foregoing sections has Cauchy's second law of motion, $t^{km} = t^{mk}$, been imposed. In terms of the response function $\mathfrak{h}$, it is equivalent to the additional condition

$$\mathfrak{h}^{k\alpha}\, x^m{}_{,\alpha} = \mathfrak{h}^{m\alpha}\, x^k{}_{,\alpha}, \tag{B}$$

as follows from (2.3) and Eq. (A) in footnote 3, p. 266. In MacCullagh's theory, on the other hand, $t^{km} = -t^{mk}$. In the present ordering of ideas, Cauchy's second law is imposed here along with the principle of material indifference, the two together being expressed by (6.1) and (6.2).

Since $C_{\gamma\delta} = C_{\delta\gamma}$, there is no loss in generality if we suppose $\mathfrak{t}^{\alpha\beta}$ to be formally symmetrized in the components $C_{\gamma\delta}$. Thus we may put (2.5) into the form

$$\begin{aligned} A_{km}{}^{\alpha\beta} &= g_{km}\,\mathfrak{t}^{\alpha\beta} + g_{kq}\,x^q{}_{,\lambda}\,\frac{\partial \mathfrak{t}^{\alpha\lambda}}{\partial C_{\mu\nu}}\,\frac{\partial C_{\mu\nu}}{\partial x^m{}_{,\beta}}, \\ &= g_{km}\,\mathfrak{t}^{\alpha\beta} + 2 g_{kp}\,g_{mq}\,x^p{}_{,\gamma}\,x^q{}_{,\delta}\,\frac{\partial \mathfrak{t}^{\alpha\gamma}}{\partial C_{\beta\delta}}. \end{aligned} \tag{6.4}$$

Let us set

$$C^{kmpq} \equiv 2\,\frac{\varrho}{\tilde{\varrho}}\,x^k{}_{,\alpha}\,x^m{}_{,\beta}\,x^p{}_{,\gamma}\,x^q{}_{,\delta}\,\frac{\partial \mathfrak{t}^{\alpha\beta}}{\partial C_{\gamma\delta}}. \tag{6.5}$$

Then

$$C^{kmpq} = C^{mkpq} = C^{kmqp}; \tag{6.6}$$

these relations of symmetry reflect the principle of material indifference.

If we now set

$$B^{kmpq} \equiv \frac{\varrho}{\tilde{\varrho}}\,x^m{}_{,\alpha}\,x^q{}_{,\beta}\,g^{kr}\,g^{ps}\,A_{rs}{}^{\alpha\beta}, \tag{6.7}$$

so that

$$A_{km}{}^{\alpha\beta} = \frac{\tilde{\varrho}}{\varrho}\,g_{kr}\,g_{ms}\,X^\alpha{}_{,p}\,X^\beta{}_{,q}\,B^{rpsq}, \tag{6.8}$$

then from (6.4) and (6.5) we see that[2]

$$B^{kmpq} = g^{kp}\,t^{mq} + C^{kmpq}. \tag{6.9}$$

Hence by (6.6) it follows that

$$\begin{aligned} B^{kmpq} - B^{mkpq} &= g^{kp}\,t^{mq} - g^{mp}\,t^{kq}, \\ B^{kmpq} - B^{kmqp} &= g^{kp}\,t^{mq} - g^{kq}\,t^{mp}, \\ B^{kmpq} - B^{mkqp} &= g^{kp}\,t^{mq} - g^{mq}\,t^{kp}. \end{aligned} \tag{6.10}$$

These identities, again, reflect the principle of material indifference.

In the classical infinitesimal theory, the tensors $A_{km}{}^{(\alpha\beta)}$, B^{kmpq}, and C^{kmpq} all reduce to the tensor of linear elasticities.

Substituting (6.8) into (2.8) yields an expression for the acoustical tensor in terms of B^{kmpq}:

$$\begin{aligned} Q^{km} &= \frac{\tilde{\varrho}}{\varrho}\,X^{(\alpha}{}_{,u}\,X^{\beta)}{}_{,v}\,x^p{}_{,\alpha}\,x^q{}_{,\beta}\,B^{kumv}\,n_p\,n_q, \\ &= \frac{1}{2}\,\frac{\tilde{\varrho}}{\varrho}\,(\delta^p_u\,\delta^q_v + \delta^q_u\,\delta^p_v)\,B^{kumv}\,n_p\,n_q, \\ &= \frac{\tilde{\varrho}}{\varrho}\,B^{kpmq}\,n_p\,n_q. \end{aligned} \tag{6.11}$$

Another consequence of the principle of material indifference is the fact that the *speeds of propagation are determined by the strain alone, independently of the local rotation.* This is most easily seen by demonstrating an alternative form of the propagation condition (2.7), expressed in terms of material rather than spatial co-ordinates. Set

$$S^\alpha_\beta \equiv X^\alpha{}_{,k}\,Q^k{}_m\,x^m{}_{,\beta}, \qquad \nu_\alpha \equiv x^k{}_{,\alpha}\,n_k. \tag{6.12}$$

---

[2] This result was given by TOUPIN & BERNSTEIN [1961, 1, Eq. (2.23)] for the case of hyperelastic materials.

Then from (6.11) and (6.9) we see that[3]

$$\begin{aligned} S^{\alpha}_{\beta} &= \frac{\tilde{\varrho}}{\varrho} X^{\alpha}{}_{,k} B^{kpsq} g_{ms} n_p n_q x^m{}_{,\beta}, \\ &= \frac{\tilde{\varrho}}{\varrho} X^{\alpha}{}_{,k} x^m{}_{,\beta} [\delta^k_m t^{pq} + g_{ms} C^{kpsq}] n_p n_q, \\ &= \left[\frac{\tilde{\varrho}}{\varrho} \delta^{\alpha}_{\beta} t^{pq} + 2 C_{\beta\lambda} x^p{}_{,\delta} x^q{}_{,\mu} \frac{\partial \mathfrak{t}^{\alpha\delta}}{\partial C_{\lambda\mu}}\right] X^{\varrho}{}_{,p} \nu_{\varrho} X^{\sigma}{}_{,q} \nu_{\sigma}, \\ &= D^{\alpha\gamma\delta}_{\beta} \nu_{\gamma} \nu_{\delta}, \end{aligned} \tag{6.13}$$

where

$$D^{\alpha\gamma\delta}_{\beta}(\boldsymbol{C}) = \delta^{\alpha}_{\beta} \mathfrak{t}^{\gamma\delta} + C_{\beta\lambda}\left(\frac{\partial \mathfrak{t}^{\alpha\gamma}}{\partial C_{\delta\lambda}} + \frac{\partial \mathfrak{t}^{\alpha\delta}}{\partial C_{\gamma\lambda}}\right). \tag{6.14}$$

Thus, for a given material, $D^{\alpha\gamma\delta}_{\beta}$ is a given function of $\boldsymbol{C}$ and hence is independent of the local rotation. From (6.13) and (6.12) we see that the propagation condition (2.7) may be written in the form

$$(S^{\alpha}_{\beta} - \tilde{\varrho}\, U^2 \delta^{\alpha}_{\beta}) A^{\beta} = 0, \tag{6.15}$$

where we may[4] choose $\boldsymbol{A}$ in such a way that

$$A^{\beta} = R^{\beta}_{k} a^k, \tag{6.16}$$

$\boldsymbol{R}$ being the local rotation. The speeds of propagation are determined from (6.15) by the cubic equation

$$\det (S^{\alpha}_{\beta} - \tilde{\varrho}\, U^2 \delta^{\alpha}_{\beta}) = 0 \tag{6.17}$$

and hence are algebraic functions of the principal invariants of the tensor $\boldsymbol{S}(\boldsymbol{C}, \boldsymbol{\nu})$. The right proper vectors of $\boldsymbol{S}$ may be calculated from those of $\boldsymbol{Q}$ by means of (6.16); thus the acoustic axes may be obtained by effecting upon the right proper vectors of $\boldsymbol{S}$ the same rotation as that carrying the principal axes of strain at $\boldsymbol{X}$ into those at $\boldsymbol{x}$.

7. Isotropic materials

For an isotropic material the function $\mathfrak{t}(\boldsymbol{C})$ is an isotropic function. By using the representation theorem for such functions it is easy to show that (6.1) may be reduced[1] to the form

$$t^k_m = f_0\, \delta^k_m + f_1\, B^k_m + f_2\, B^k_p B^p_m, \tag{7.1}$$

where[2]

$$B^{km} \equiv g^{\alpha\beta} x^k{}_{,\alpha} x^m{}_{,\beta}, \tag{7.2}$$

and where the response coefficients f_0, f_1, f_2 are functions of the principal invariants I, II, III of $\boldsymbol{B}$. Cauchy's first law of motion (2.4) may be written in the

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[3] This analysis generalizes and simplifies that of Toupin & Bernstein [1961, 1, §4]. It employs the identity

$$t^{qp} = \frac{\varrho}{\tilde{\varrho}} x^q{}_{,\gamma} x^p{}_{,\delta} \mathfrak{t}^{\gamma\delta}(\boldsymbol{C}), \tag{C}$$

which follows from (6.1) and Eq. (A) in footnote 3, p. 266. The result is equivalent to one stated and proved in a different way, again for the hyperelastic case, by Tolotti [1943, §1, p. 1]: The form of the quadric of polarization is independent of the local rotation.

[4] This possibility follows by decomposing the deformation into a rotation and a pure stretch; *cf.* CFT, §37.

[1] *E.g.*, §303 of CFT.

[2] $\boldsymbol{B}$ is the tensor denoted by $\boldsymbol{c}^{-1}$ in §29 of CFT.

more familiar form [CFT, Eq. (205.2)]

$$t^{km}{}_{,m} + \varrho f^k = \varrho \ddot{x}^k, \tag{7.3}$$

where ϱ is the density in the deformed material. By putting (7.1) into (7.3) we see that[3]

$$T^{km}{}_{pq} B^{pq}{}_{,m} + \varrho f^k = \varrho \ddot{x}^k, \tag{7.4}$$

where[4]

$$\begin{aligned} T^{km}{}_{pq} &\equiv \frac{\partial t^{km}}{\partial B^{pq}} = T^{mk}{}_{pq} = T^{km}{}_{qp}, \\ &= \tfrac{1}{2} f_1 (\delta^k_p \delta^m_q + \delta^m_p \delta^k_q) + \\ &\quad + \tfrac{1}{2} f_2 (\delta^k_p B^m_q + \delta^k_q B^m_p + \delta^m_p B^k_q + \delta^m_q B^k_p) + \\ &\quad + g^{km} \left[\frac{\partial f_0}{\partial I} g_{pq} + \frac{\partial f_0}{\partial II} (I g_{pq} - B_{pq}) + III \frac{\partial f_0}{\partial III} \overset{-1}{B}_{pq} \right] + \\ &\quad + B^{km} \left[\frac{\partial f_1}{\partial I} g_{pq} + \frac{\partial f_1}{\partial II} (I g_{pq} - B_{pq}) + III \frac{\partial f_1}{\partial III} \overset{-1}{B}_{pq} \right] + \\ &\quad + B^k_r B^{rm} \left[\frac{\partial f_2}{\partial I} g_{pq} + \frac{\partial f_2}{\partial II} (I g_{pq} - B_{pq}) + III \frac{\partial f_2}{\partial III} \overset{-1}{B}_{pq} \right], \end{aligned} \tag{7.5}$$

and, by (7.2),

$$B^{pq}{}_{,m} = X^\alpha{}_{,m} g^{\beta\gamma} (x^p{}_{,\gamma} x^q{}_{;\alpha\beta} + x^q{}_{,\gamma} x^p{}_{;\alpha\beta}). \tag{7.6}$$

If we calculate the jump of (7.4) at a singular surface of second order, by using (7.6) and (2.1) we again obtain the propagation condition (2.7), where $\boldsymbol{Q}$ now has the following explicit form:

$$Q^k_m = 2 \frac{\tilde{\varrho}}{\varrho} T^{kp}{}_{mq} B^{qs} n_p n_s. \tag{7.7}$$

By (7.5), $T^{kp}{}_{mq}$ is a function of $\boldsymbol{B}$ only; hence, for given $\boldsymbol{n}$, the acoustical tensor $\boldsymbol{Q}$ is a function of $\boldsymbol{B}$ only. Therefore, as is only natural, *in an isotropic medium not only the speeds of propagation but also the acoustic axes for waves travelling in a given direction are determined by the strain alone.*

A wave travelling down a principal axis of stress and strain in an isotropic medium may be called a *principal wave.* The results to be expected in this special case have been described in §1. To render explicit the statements made there, let $\boldsymbol{n}_1$ be a proper vector of $\boldsymbol{B}$, so that $B^r_s n^s_1 = v_1^2 n^r_1$, $\overset{-1}{B}{}^r_s n^s_1 = n^r_1 / v_1^2$, where v_1 is the stretch in the direction of the principal axis corresponding to $\boldsymbol{n}_1$. Substituting (7.5) into (7.7) yields at once the following explicit form for the acoustical tensor[5]:

$$\begin{aligned} \frac{\varrho}{\tilde{\varrho}} v_1^{-2} Q^k_m(\boldsymbol{n}_1) &= (f_1 + v_1^2 f_2) \delta^k_m + f_2 B^k_m + \\ &\quad + 2 n^k_1 n^1_m \left\{ \frac{1}{2} f_1 + v_1^2 f_2 + \sum_{\Gamma=0}^{2} v_1^{2\Gamma} \left[\frac{\partial f_\Gamma}{\partial I} + (v_2^2 + v_3^2) \frac{\partial f_\Gamma}{\partial II} + v_2^2 v_3^2 \frac{\partial f_\Gamma}{\partial III} \right] \right\}. \end{aligned} \tag{7.8}$$

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[3] The approach here is that introduced by ERICKSEN [1953, 1] for more special circumstances.

[4] To effect the calculation, use Eqs. App. (38.16) of CFT.

[5] In effect, this tensor may be calculated (in the hyperelastic case) by combining Eqs. (4.8), (4.7), and (3.10) of the paper by HAYES & RIVLIN [1961, 2], but, perhaps hindered by an exuberance of formalism, they did not state the simple and conclusive results derived here. Earlier TOLOTTI [1943, Eqs. (18′) (19′)] had virtually calculated $\boldsymbol{Q}(\boldsymbol{n})$ for isotropic hyperelastic materials but, again apparently because of obscurity of notation and method, did not conclude anything except the independence of the acoustic axes from the local rotation.

Since the remaining principal axes of $\boldsymbol{B}$ are normal to $\boldsymbol{n}_1$, we see by inspection of (7.8) that they are also principal axes of $\underset{\sim}{\boldsymbol{Q}}(\boldsymbol{n}_1)$; that is, *the acoustic axes for principal waves coincide with the principal axes.* In particular, *every principal wave is either longitudinal or transverse.*

By using (2.10) we may now calculate the speeds of propagation exactly and explicitly[6]. Setting $\boldsymbol{a}=\boldsymbol{n}_1$, we obtain the squared speed U_{11}^2 of longitudinal waves along the principal axis in which the stretch is v_1:

$$\frac{\varrho U_{11}^2}{2v_1^2} = f_1 + 2v_1^2 f_2 + \sum_{\Gamma=0}^{2} v_1^{2\Gamma}\left[\frac{\partial f_\Gamma}{\partial I} + (v_2^2+v_3^2)\frac{\partial f_\Gamma}{\partial II} + v_2^2 v_3^2\frac{\partial f_\Gamma}{\partial III}\right], \tag{7.9}$$

while for transverse waves with amplitude parallel to $\boldsymbol{n}_2$ the squared speed U_{12}^2 is given by

$$\frac{\varrho U_{12}^2}{v_1^2} = f_1 + (v_1^2+v_2^2)\, f_2. \tag{7.10}$$

The formulae (7.9) and (7.10) give the ***explicit and exact speeds of propagation of principal waves.*** ERICKSEN[7] has pointed out that they assume a simpler form if expressed in terms of the relations giving the principal stresses t_a as functions of the principal stretches:

$$t_1 = f_0 + f_1 v_1^2 + f_2 v_1^4, \tag{7.11}$$

as follows from (7.1). Since

$$\frac{\partial f_\Gamma}{\partial v_1^2} = \frac{\partial f_\Gamma}{\partial I} + (v_2^2+v_3^2)\frac{\partial f_\Gamma}{\partial II} + v_2^2 v_3^2\frac{\partial f_\Gamma}{\partial III}, \tag{7.12}$$

from (7.11) we see that the right-hand side of (7.9) is equal to $\partial t_1/\partial v_1^2$. Hence[8]

$$\varrho U_{11}^2 = \frac{\partial t_1}{\partial \log v_1}, \tag{7.13}$$

while from (7.10) it follows at once that

$$\frac{\varrho U_{12}^2}{v_1^2} = \frac{t_1 - t_2}{v_1^2 - v_2^2}. \tag{7.14}$$

Since these elegant formulae do not make direct use of the response coefficients f_Γ, they give insight into the general nature of wave propagation in all kinds of elastic materials.

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[6] These results generalize a long sequence of special formulae for various kinds of materials in various circumstances. Some of the relevant literature, which is mainly physical, is cited by TOUPIN & BERNSTEIN [1961, 1].

The formula (7.10) is similar to one given by ERICKSEN [1953, 1, Eq. (4.6)] for incompressible hyperelastic materials. The results above, however, do not seem to be obvious from his, because the condition of incompressibility by itself renders all waves transversal, and thus for incompressible materials the whole picture of wave propagation is not really representative of the general case.

[7] Letter dated July 13, 1961.

[8] Here t_1 is regarded as a function of the three *independent* variables $\log v_1$, $\log v_2$ and $\log v_3$. For special families of deformations a result of different form will follow. For example, if only deformations such that $v_1=v_2=v_3$ are considered, we obtain a result of the form (8.4) below.

By using (7.10) we see that

$$\frac{\varrho}{v_1^2}\,(U_{12}^2 - U_{13}^2) = (v_2^2 - v_3^2)\,f_2. \tag{7.15}$$

From this formula follows the ***fundamental theorem on transverse principal waves:*** *In an isotropic elastic material such that $f_2 = 0$ for the strain in question, both kinds of transverse waves that may travel down a given principal axis of strain have the same absolute speed of propagation; if, however, $f_2 \neq 0$, then these two kinds of transverse waves travel at the same absolute speed if and only if the corresponding principal stretches are equal; and in this latter case, any transverse amplitude is possible, and all these transverse waves have the same absolute speeds of propagation.* From (7.15), moreover, we may interpret the sign of the function f_2. Excluding the case when $v_2 = v_3$, we see that *if $f_2 < 0$, transverse waves with amplitudes parallel to the axis of lesser transverse stretch travel at greater absolute speed than the others; if $f_2 > 0$, at lesser*[9].

From (7.9) it is clear that $U_{11}^2 = U_{22}^2$ if $v_1 = v_2$. A converse condition is immediate from (7.13), leading to the ***fundamental theorem on longitudinal principal waves:*** *In an isotropic elastic material, the squared speeds of longitudinal waves in the directions of two equal principal stretches are equal; conversely, if $U_{11}^2 = U_{22}^2$, it is necessary that*[10]

$$\frac{\partial t_1}{\partial \log v_1} = \frac{\partial t_2}{\partial \log v_2}. \tag{7.16}$$

These results show that the state of affairs in the linearized theory, where all transverse waves travel at one absolute speed and all longitudinal waves at another, is not at all typical for elastic materials.

8. Isotropic materials subject to hydrostatic pressure

Only the case of hydrostatic stress partakes of the simplicity to which the linearized theory has accustomed us. Indeed, if $v_1 = v_2 = v_3 = v$, then every direction is a principal axis of stress and strain, so that all waves are principal waves, while (7.9) and (7.10) show that $U_{11}^2 = U_{22}^2 = U_{33}^2 = U_{\parallel}^2$, say, and $U_{12}^2 = U_{21}^2 = \cdots = U_{32}^2 = U_{\perp}^2$, say. That is, *in a state of hydrostatic pressure, longitudinal and transverse waves may propagate in any direction*[1]*; at each point, all longitudinal waves have the same absolute speed, and all transverse waves have the same absolute speed.* To calculate these speeds, we note that since $I = 3v^2$, $II = 3v^4$, $III = v^6$, it follows that

$$\frac{\partial}{\partial I} + 2\frac{\partial}{\partial II} + \frac{\partial}{\partial III} = \frac{1}{3}\frac{d}{dv} \tag{8.1}$$

for the class of deformations being considered. Moreover, by (7.1), $t_m^k = -p\,\delta_m^k$ where $p = p(\varrho)$. From (7.9) and (7.10) it is now easy to show that

$$U_{\parallel}^2 = \frac{4}{3}\,\frac{v^5}{\tilde{\varrho}}\,(f_1 + 2v^2 f_2) + \frac{dp}{d\varrho}, \tag{8.2}$$

$$U_{\perp}^2 = \frac{v^5}{\tilde{\varrho}}\,(f_1 + 2v^2 f_2). \tag{8.3}$$

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[9] For rubbers $f_2 < 0$ according to the results of experiments. The sign of the more frequently used coefficient h_{-1}, which is introduced in § 11, is the same as that of f_2.

[10] By calculating $U_{11}^2 - U_{22}^2$ from (7.9) it is a trivial matter to express this condition in terms of the response coefficients f_Γ, but the resulting formula is too elaborate to be useful.

[1] Recall the caution expressed in § 5.

By eliminating $f_1 + 2v^2 f_2$ between these two results, we derive the following not inelegant *universal relation* between the two wave speeds and the bulk modulus in any state of hydrostatic pressure:

$$U_\parallel^2 = \frac{4}{3} U_\perp^2 + \frac{dp}{d\varrho}. \tag{8.4}$$

Two special cases are already familiar. First, for an ideal fluid we have $f_1 = f_2 = 0$, while $f_0 = g(III) = -p(\varrho)$, say. By (8.3), $U_\perp = 0$ in this case[2], while (8.4) reduces to the classical formula for the speed of sound in fluids. Second, in the linearized theory of elasticity, (8.4) reduces to a trivial identity connecting the two squared wave-speeds, *viz* $(\lambda + 2\mu)/\varrho$, μ/ϱ, with the bulk modulus, $(\lambda + \frac{2}{3}\mu)/\varrho$.

It is reasonable to assume that $dp/d\varrho > 0$ in all circumstances. Thus from (8.4) we see that

$$U_\parallel^2 > \tfrac{4}{3} U_\perp^2 \tag{8.5}$$

for any state of hydrostatic pressure in any elastic material. In particular, *longitudinal waves always propagate faster than transverse waves*, provided the latter can exist at all. Whether or not $U_\perp$ is real is a more delicate matter. Some conditions of reality are discussed in §10. For the time being, however, we content ourselves with interpreting (8.4) as follows: *In a state of hydrostatic stress, if transverse waves may propagate at all, so may longitudinal waves, and then*

$$U_\parallel^2 > \frac{dp}{d\varrho}; \tag{8.6}$$

that is, if transverse waves may propagate, the squared speed of longitudinal waves is always greater than the squared wave-speed according to the hydrodynamics of an ideal fluid with the same law of compression. If, however, $U_\perp^2 < 0$, then

$$U_\parallel^2 < \frac{dp}{d\varrho}. \tag{8.7}$$

Therefore, if neither material transverse singularities nor transverse waves are possible, the squared speed of longitudinal waves is either less than that of sound waves in a fluid with the same law of compression, or is zero, or is imaginary. The case when $U_\parallel^2 < 0$ cannot be excluded *a priori*.

The foregoing results on the speeds of waves in elastic materials subject to great hydrostatic pressure may be not devoid of interest to seismologists.

9. Determination of the stress-strain relations from wave speeds

When the form of the stress-strain relations is known, formulae such as (7.9) and (7.10) give unique values for the squared speeds of the 9 possible kinds of principal waves. Conversely, if these 9 squared speeds are known as functions of the three principal stretches v_a, as for example from the results of experiments, then the three response coefficients f_0, f_1, and f_2 are uniquely determined.

Indeed, they are overdetermined. From (7.9) and (7.10) we may derive a sequence of *universal relations* connecting the wave speeds, relations which must hold in any isotropic elastic material regardless of the forms of f_0, f_1, and f_2.

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[2] Thus transverse singular surfaces of second or higher order cannot propagate in an ideal fluid. Other remarks on this case are given in footnote 4, p. 281, below.

Only if these relations are satisfied can there exist functions f_0, f_1, and f_2 such that (7.9) and (7.10) hold. Thus these universal relations serve as *conditions of compatibility* for measured values of the wave speeds: Unless these relations be satisfied by the data, it is not consistent with the theory of isotropic elasticity[1].

Compatibility of transverse waves. From (7.10) we derive at once 2 sets of 3 universal relations:

$$\frac{U_{12}^2}{v_1^2}=\frac{U_{21}^2}{v_2^2}, \qquad \frac{U_{23}^2}{v_2^2}=\frac{U_{32}^2}{v_3^2}, \qquad \frac{U_{31}^2}{v_3^2}=\frac{U_{13}^2}{v_1^2}, \tag{9.1}$$

$$\begin{gathered}\frac{1}{v_1^2-v_3^2}\left(\frac{U_{12}^2}{v_1^2}-\frac{U_{23}^2}{v_2^2}\right)=\frac{1}{v_2^2-v_1^2}\left(\frac{U_{23}^2}{v_2^2}-\frac{U_{31}^2}{v_3^2}\right)=\frac{1}{v_3^2-v_2^2}\left(\frac{U_{31}^2}{v_3^2}-\frac{U_{12}^2}{v_1^2}\right),\\ \equiv\varphi(v_1,v_2,v_3),\end{gathered} \tag{9.2}$$

where φ is a symmetric function of its arguments. Of these 6 conditions, the first 3 are independent, but from any 1 of the last 3 the other 2 may be derived by use of the first set, so in all there are 4 independent conditions. If these are satisfied, the functions f_1 and f_2 are uniquely determined by relations of the type (7.10). For example,

$$\begin{aligned} f_2&=\varrho\,\varphi=\varrho\,\frac{U_{12}^2-U_{13}^2}{v_1^2(v_2^2-v_3^2)},\\ f_1&=\frac{1}{3}\varrho\left[\frac{U_{12}^2}{v_1^2}+\frac{U_{23}^2}{v_2^2}+\frac{U_{31}^2}{v_3^2}-2I\,\varphi\right],\\ &=\varrho\,\frac{(v_2^2-I)\,U_{12}^2-(v_3^2-I)\,U_{13}^2}{v_1^2(v_2^2-v_3^2)}. \end{aligned} \tag{9.3}$$

In each case the second form follows from the first by use of the identities (9.1) and (9.2). From the form of the results we see that *in an isotropic elastic material, the response coefficients f_1 and f_2 are uniquely determined by the speeds of the two kinds of transverse principal waves corresponding to any one principal axis.* The experimenter willing to take compatibility for granted may content himself with measuring U_{12}^2 and U_{13}^2 as functions of v_1, v_2, v_3.

Compatibility of longitudinal waves. Let us assume that the conditions of compatibility for the transverse principal waves are satisfied, so that f_1 and f_2 are uniquely determined. We now consider the compatibility of the speeds of the 3 longitudinal waves, from which the single function f_0 is to be determined.

If we set

$$A_1=\frac{1}{2}\varrho\,\frac{U_{11}^2}{v_1^2}-f_1-2v_1^2 f_2-\sum_{\Gamma=1}^{2} v_1^{2\Gamma}\left[\frac{\partial f_\Gamma}{\partial I}+(v_2^2+v_3^2)\,\frac{\partial f_\Gamma}{\partial II}+v_2^2 v_3^2\,\frac{\partial f_\Gamma}{\partial III}\right], \tag{9.4}$$

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[1] TOUPIN & BERNSTEIN [1961, 1, §4] were the first to consider problems of this kind. They derived necessary and sufficient conditions for the linearized theory of hyperelastic materials. Earlier HUGHES & KELLY [1953, 2, p. 1146], after obtaining more than sufficient formulae for determining the second-order elasticities of isotropic hyperelastic materials, had remarked that "any additional measurements give checks on the theory".

The results I present above for the finite theory are sufficient only as regards the nine kinds of principal waves. If waves travelling in other directions are to be considered, additional conditions of compatibility must be satisfied.

etc., then because of (9.3) we can regard the 3 quantities A_a as explicitly given functions of U_{12}^2, U_{13}^2, U_{11}^2, U_{22}^2, U_{33}^2, v_1, v_2, v_3, and we are to find f_0 from 3 relations of the type

$$A_1 = \frac{\partial f_0}{\partial I} + (v_2^2 + v_3^2)\frac{\partial f_0}{\partial II} + v_2^2 v_3^2 \frac{\partial f_0}{\partial III}. \tag{9.5}$$

If the three principal stretches v_a are distinct, we may solve (9.5) as a linear system, obtaining the following results:

$$\begin{aligned}
\frac{\partial f_0}{\partial I} &= \frac{A_1 v_1^4 (v_2^2 - v_3^2) + A_2 v_2^4 (v_3^2 - v_1^2) + A_3 v_3^4 (v_1^2 - v_2^2)}{(v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_3^2 - v_1^2)} \equiv B_1, \\
\frac{\partial f_0}{\partial II} &= -\frac{A_1 v_1^2 (v_2^2 - v_3^2) + A_2 v_2^2 (v_3^2 - v_1^2) + A_3 v_3^2 (v_1^2 - v_2^2)}{(v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_3^2 - v_1^2)} \equiv B_2, \\
\frac{\partial f_0}{\partial III} &= \frac{A_1 (v_2^2 - v_3^2) + A_2 (v_3^2 - v_1^2) + A_3 (v_1^2 - v_2^2)}{(v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_3^2 - v_1^2)} \equiv B_3.
\end{aligned} \tag{9.6}$$

The quantities B_a are known functions of v_1, v_2, v_3; they must be symmetric functions and hence equal to functions of I, II, III. In order that f_0 may be determined from the system (9.6), the functions B_a must satisfy the following conditions of integrability:

$$\frac{\partial B_1}{\partial II} = \frac{\partial B_2}{\partial I}, \qquad \frac{\partial B_2}{\partial III} = \frac{\partial B_3}{\partial II}, \qquad \frac{\partial B_1}{\partial III} = \frac{\partial B_3}{\partial I}. \tag{9.7}$$

If these conditions are satisfied, then

$$f_0 = \int (B_1\, dI + B_2\, dII + B_3\, dIII). \tag{9.8}$$

A summary of these results is given by the ***compatibility theorem:*** *Let the 9 squared speeds of propagation of principal waves be known functions of the principal stretches. This data is consistent with the theory of isotropic elastic materials if and only if:*

(a) *The quantities* φ, B_1, B_2, B_3, *as defined by* $(9.2)_3$ *and* (9.6), *are symmetric functions of the principal stretches.*

(b) *The* 4 *algebraic conditions* (9.1) *and* $(9.2)_1$ *and the* 3 *differential conditions* (9.7) *are satisfied.*

If conditions (a) *and* (b) *hold, the data determines a unique form for the stress-strain relations by means of* (9.3) *and* (9.8).

10. Conditions that the waves be real

As yet, we have not taken up the question of whether the speeds and corresponding amplitudes are real. Only real waves have physical significance. The results so far stated are carefully phrased so as to refer, for the most part, only to necessary conditions. In the generality maintained thus far, *it is possible that no waves can exist.* For $\underset{\sim}{Q}(\boldsymbol{n})$, being an arbitrary 3×3 matrix, may have but one real right proper vector, and the corresponding proper number may be negative, leading to a purely imaginary value of U.

Various inequalities have been proposed so as to insure that the response function $\mathfrak{h}_k^{\alpha}$ in (2.3) describe a material not altogether unreasonable for physical applications[1]. All of these turn out to have some bearing on the propagation of waves, and in this context we are now in a position to read off results more general than any heretofore known.

First, for the hyperelastic case HADAMARD [1903, p. 269] gave a definition of stability from which he proved that in a stable state all speeds of propagation are real. I shall not discuss this condition here[2] because it is not clear how to extend HADAMARD'S definition of stability to general elastic materials, when there need be no strain energy.

In the hyperelastic case, however, HADAMARD'S criterion implies that the equations of equilibrium are *strongly elliptic*, and a condition to this end we can easily formulate and discuss in full generality from (2.6); namely, the *S-E condition* is

$$A_{(km)}{}^{(\alpha\beta)}\, l^k\, l^m\, \lambda_\alpha\, \lambda_\beta > 0 \tag{10.1}$$

for arbitrary vectors $\boldsymbol{l}$ and $\boldsymbol{\lambda}$. From (2.8) we see that an equivalent condition is

$$Q_{(km)}(\boldsymbol{n})\, l^k\, l^m > 0 \tag{10.2}$$

for arbitrary $\boldsymbol{n}$ and $\boldsymbol{l}$. That is, *the S-E condition is necessary and sufficient that the symmetric part of the acoustical tensor be positive-definite, for every direction of propagation.* From (2.10) it thus follows that $U^2 > 0$ if $\boldsymbol{a}$ is real. That is, *if the S-E condition holds, the squared speed of propagation corresponding to every real acoustic axis is positive.* In particular, since for given $\boldsymbol{n}$ there is at least one real amplitude $\boldsymbol{a}$, *the S-E condition implies that for any direction of propagation there exists at least one real amplitude with positive speed.*

Since, however, the *S-E* condition restricts only the symmetric part of $\boldsymbol{Q}(\boldsymbol{n})$, it cannot guarantee that in general there are three real acoustic axes. In the important special case of principal waves in an isotropic material, we have exhibited three real axes. By the results in §6, then, *the S-E condition implies that the squared speeds of all principal waves in an isotropic material are positive.*

The foregoing results, except for the last one, which refers to isotropic materials, are independent of the principle of material indifference. When this principle is satisfied, from (6.8) we see that (10.1) may be expressed in the following equivalent form:

$$B^{kpmq}\, l_k\, l_m\, L_p\, L_q > 0 \tag{10.3}$$

for arbitrary vectors $\boldsymbol{l}$ and $\boldsymbol{L}$. A formally similar but in fact different condition is

$$B^{kpmq}\, S_{kp}\, S_{mq} > 0 \tag{10.4}$$

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[1] A review of this question, which I proposed in a lecture [1956] before the Gesellschaft für angewandte Mathematik und Mechanik in Berlin in 1955, is given by TOUPIN [1961, 3].

[2] It must be added, however, that the criticism by SIGNORINI and TOLOTTI [1943] entirely misses the point of HADAMARD'S work by omitting any mention of the question of stability. As shown in the text above, it is entirely obvious that for fully general stress-strain relations there need be no possibility of real waves. Moreover, the very special stress-strain relations derived by TOLOTTI [1943, Eq. (3)] do not exhaust the possible isotropic hyperelastic materials in which all speeds of propagation are real, for TOLOTTI imposes at the outset the unduly restrictive additional condition that one acoustical axis be longitudinal for any direction of propagation, while transverse waves may have *any* perpendicular amplitude. As to be expected, he succeeds in showing that scarcely any materials are of this kind.

HADAMARD'S analysis of stability was questioned by DUHEM [1905, Ch. III, § IV].

for all symmetric non-vanishing tensors $\boldsymbol{S}$. For hyperelastic materials, a condition of this type is equivalent to one proposed by COLEMAN & NOLL[3] on the basis of a compelling thermodynamic argument, supported by important corollaries in statics. We shall call (10.4) *the generalized C-N condition.* Unlike the *S-E* condition, it cannot be expressed entirely in terms of the acoustical tensor.

It is difficult to draw general conclusions from (10.4). From examples[4] it is known that neither the generalized *C-N* condition nor the *S-E* condition implies the other for all materials and all strains. In the special case when we put $S_{kp} = n_k n_p$, by using (6.11) we derive at once the necessary condition

$$Q^{km}(\boldsymbol{n})\, n_k n_m > 0. \tag{10.5}$$

In general, $\boldsymbol{n}$ is not a right proper vector of $\boldsymbol{Q}(\boldsymbol{n})$; if it is, from (10.5) and (2.10) we see that the corresponding speed of propagation is real. That is, *the generalized C-N condition insures that if a longitudinal amplitude is possible, the corresponding squared speed is positive.*

In the isotropic case, ERICKSEN's formulae (7.13) and (7.14) make possible the statement of simple and immediate criteria, which may be summarized in the ***fundamental theorem on the speeds of principal waves:*** *In an isotropic material the squared speeds of the principal longitudinal waves are positive if and only if each principal tension is an increasing function of the corresponding principal stretch when the other principal stretches are held constant; the squared speeds of the principal transverse waves are positive if and only if the greater principal tension occurs always in the direction of the greater principal stretch.*

As has already been shown, the *S-E* condition implies that all squared wave speeds are positive; hence it implies the two statical conditions just stated. The first of these is implied also by the generalized *C-N* condition; to verify this fact,

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[3] For the connection with the work of COLEMAN & NOLL [1959, 1], the reader is referred to the paper of TOUPIN & BERNSTEIN [1961, 1, §3]. As explained there, the requirements of COLEMAN & NOLL allow equality to replace inequality in (10.4) for certain exceptional strains, but in the interest of simpler statements of theorems we do not consider these.

[4] TOUPIN & BERNSTEIN [1960, 1, §3] construct a simple example of an isotropic hyperelastic solid which when subject to hydrostatic pressures within a certain range satisfies the *C-N* condition but does not satisfy the *S-E* condition.

As they remark, an ideal fluid for which $p > 0$ and $dp/d\varrho > 0$ serves as an example of a material in which the *C-N* condition is satisfied for all strains but the *S-E* condition for none. To verify this fact from the present point of view it suffices to note that $f_0 = g(III) = -p(\varrho)$, $f_1 = f_2 = 0$, so that (7.7) yields

$$Q^k_m(\boldsymbol{n}) = 2\,\frac{\tilde{\varrho}}{\varrho}\, III\, \frac{\partial g}{\partial III}\, n^k n_m = \tilde{\varrho}\, \frac{dp}{d\varrho}\, n^k n_m. \tag{D}$$

As in all isotropic materials subject to hydrostatic pressure, any transverse amplitude is possible, *but here the corresponding speed of propagation is zero.* That is, in the notation of §8, $U_\perp = 0$. From (2.1) and (3.1) it follows that *in an ideal fluid, the only possible transverse singular surfaces of order greater than* 1 *are material surfaces across which only certain purely spatial derivatives such as* $x^k{}_{,\alpha\beta}$, $x^k{}_{,\alpha\beta\gamma}$, etc. *suffer discontinuities* [*cf.* CFT, §190]. The *C-N* condition, as was shown in the researches of COLEMAN & NOLL, is sufficient that $dp/d\varrho > 0$ and hence that $U_\parallel^2 > 0$, by (8.4). Since $U_\perp = 0$, the *S-E* condition can never be satisfied. Conversely, the condition that the squared speeds of longitudinal waves be positive, since it is equivalent to $dp/d\varrho > 0$, is weaker than the *C-N* condition, which implies that $p > 0$ also.

we need only notice that, as was shown in §7, a longitudinal amplitude is always possible for a principal wave in an isotropic material, so that the italicized statement following (10.5) becomes applicable. Thus *the generalized C-N condition implies that the squared speeds of all principal longitudinal waves in an isotropic material are positive*[5].

The generalized *C-N* condition is not so intimately connected with transverse waves. The second statical condition in the last theorem, namely, that the greater principal tension occurs in the direction of the greater principal stretch, was proposed and explored by BAKER & ERICKSEN [1954] and may be called the *B-E condition*[6]. Merely rephrasing a result already proved, we may assert that *the B-E condition is necessary and sufficient that the squared speeds of all principal transverse waves be positive*[7]. It is easy to show directly that for isotropic hyperelastic[8] materials the *C-N* condition implies the *B-E* condition if all principal tensions are positive. Hence *the C-N condition implies that in a state of pure tension in an isotropic hyperelastic material, the squared speeds of all principal waves are positive.*

11. Waves according to the second-order theory of elasticity for isotropic materials*

We now apply the foregoing results to the determination of second-order elastic moduli from the speeds of waves. To this end we write (7.1) in the form[1]

$$t^k_m = h_0\, \delta^k_m + h_1\, B^k_m + h_{-1}\, \overset{-1}{B}{}^k_m, \tag{11.1}$$

where the coefficients $h_\Gamma(I, II, III)$ are related as follows to the coefficients f_Γ used previously:

$$f_0 = h_0 + \frac{II}{III} h_{-1}, \quad f_1 = h_1 - \frac{I}{III} h_{-1}, \quad f_2 = \frac{I}{III} h_{-1}. \tag{11.2}$$

The classical tensor $\boldsymbol{E}$ for measuring finite strain is given by

$$E^\alpha_\beta \equiv \tfrac{1}{2}(C^\alpha_\beta - \delta^\alpha_\beta). \tag{11.3}$$

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* A treatment of this subject from a slightly different viewpoint will appear in the proceedings of the Symposium on Second-Order Effects in Elasticity, Plasticity and Fluid Dynamics, to be held at Haifa next Easter.

[5] This theorem is due to TOUPIN [1961, 3, Th. XII]. While his argument goes along correct lines, the undetermined form for $\underset{\sim}{Q}(\boldsymbol{n})$ he in effect assumes as starting point is not sufficiently general, as may be seen from the explicit formula (7.8), above.

[6] In fact BAKER & ERICKSEN allow inequality to be replaced by equality if $v_a = v_b$, but for ease of stating the result we leave this possibility out of account.

[7] Hence follows as a trivial corollary a result apparently first published, and only for hyperelastic materials, by HAYES & RIVLIN [1961, 2]: *The S-E condition implies the B-E condition.* This fact is easily proved directly from the definitions, without reference to the theory of waves.

The theorem in the text above is close to one proved by ERICKSEN [1953, 1, §4] for incompressible hyperelastic materials.

[8] I believe this result holds for all isotropic elastic materials but have not yet succeeded in constructing a proof.

[1] For the purposes of the present paper it would be easier to remain with the coefficients f_Γ, but the coefficients h_Γ are those employed in most of the modern researches on other problems in elasticity.

Hence the principal invariants of $\boldsymbol{E}$ are related to those of $\boldsymbol{B}$ by the following identities [CFT, Eqs. (31.5)]:

$$\begin{aligned} I_{\boldsymbol{E}} &= \tfrac{1}{2}(I-3), & I &= 3+2I_{\boldsymbol{E}}, \\ II_{\boldsymbol{E}} &= \tfrac{1}{4}(II-2I+3), & II &= 3+4I_{\boldsymbol{E}}+4II_{\boldsymbol{E}}, \\ III_{\boldsymbol{E}} &= \tfrac{1}{8}(III-II+I-1), & III &= 1+2I_{\boldsymbol{E}}+4II_{\boldsymbol{E}}+8III_{\boldsymbol{E}}. \end{aligned} \tag{11.4}$$

Let the response coefficients h_Γ be expanded in power series in the invariants of $\boldsymbol{E}$:

$$h_\Gamma/\mu = \alpha_{\Gamma 0} + \alpha_{\Gamma 1} I_{\boldsymbol{E}} + \alpha_{\Gamma 2} I_{\boldsymbol{E}}^2 + \alpha_{\Gamma 3} II_{\boldsymbol{E}} + \cdots. \tag{11.5}$$

The extensions δ_a are defined in terms of the stretches v_a as follows:

$$v_a = 1 + \delta_a. \tag{11.6}$$

Our objective is to express the wave speeds approximately as linear functions of the δ_a, with coeffcients determined by the $\alpha_{\Gamma\Delta}$ occurring in the assumed expansions (11.5). It turns out that in the results the $\alpha_{\Gamma\Delta}$ occur only in the following combinations[2]:

$$\begin{aligned} \frac{\lambda}{\mu} &\equiv \alpha_1 \equiv \alpha_{01} + \alpha_{11} + \alpha_{-11}, \\ 1 &\equiv \alpha_{10} - \alpha_{-10}, \\ \alpha_3 &\equiv \alpha_{02} + \alpha_{12} + \alpha_{-12}, \\ \alpha_4 &\equiv \alpha_{03} + \alpha_{13} + \alpha_{-13}, \\ \alpha_5 &\equiv 2(\alpha_{11} - \alpha_{-11}), \\ \alpha_6 &\equiv 4\alpha_{-10}, \end{aligned} \tag{11.7}$$

where λ and μ are the classical linear moduli.

Let us introduce the notation $\doteq$ to indicate equality to within an error which is $O(\delta_1^2+\delta_2^2+\delta_3^2)$ as $\delta_1^2+\delta_2^2+\delta_3^2 \to 0$. Let ϑ be the linear approximation to the cubical dilation:

$$\vartheta \equiv \delta_1 + \delta_2 + \delta_3. \tag{11.8}$$

Then

$$I_{\tilde{\boldsymbol{E}}} \doteq \vartheta. \tag{11.9}$$

From (7.10), by use of (11.5) and (11.7), we see that

$$\begin{aligned} \frac{\varrho U_{12}^2}{\mu} &\doteq \frac{1}{\mu}(1+2\delta_1)\,h_1 - \frac{1}{\mu}(1-2\delta_2)\,h_{-1}, \\ &\doteq (1+2\delta_1)(\alpha_{10}+\alpha_{11}\vartheta) - (1-2\delta_1)(\alpha_{-10}+\alpha_{-11}\vartheta), \\ &= (1+2\delta_1)(\alpha_{10}-\alpha_{-10}) + (\alpha_{11}-\alpha_{-11})\vartheta + 2\alpha_{-10}(\delta_1+\delta_2), \\ &= 1 + 2\delta_1 + \tfrac{1}{2}\alpha_5\vartheta + \tfrac{1}{2}\alpha_6(\delta_1+\delta_2). \end{aligned} \tag{11.10}$$

Since

$$\varrho_0 \doteq \varrho(1+\vartheta), \tag{11.11}$$

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[2] To the same order of approximation, the stress-strain relations are of the form

$$t_m^k/\mu = \alpha_1 I_{\boldsymbol{E}}\,\delta_m^k + 2E_m^k + (\alpha_3 I_{\tilde{\boldsymbol{E}}}^2 + \alpha_4 II_{\tilde{\boldsymbol{E}}})\,\delta_m^k + \alpha_5 I_{\tilde{\boldsymbol{E}}}\,\tilde{E}_m^k + \alpha_6\,\tilde{E}_p^k\,\tilde{E}_k^p, \tag{E}$$

where $\tilde{\boldsymbol{E}}$ is the classical linear tensor of infinitesimal strain, but this fact is not used in the analysis.

we may express (11.10) in the alternative form

$$\frac{\varrho_0 U_{12}^2}{\mu} \doteq 1 + 2\delta_1 + \left(1 + \frac{1}{2}\alpha_5\right)\vartheta + \frac{1}{2}\alpha_6(\delta_1 + \delta_2), \tag{11.12}$$

giving the *second-order speeds of transverse principal waves.*

It is not so easy to get the corresponding results for longitudinal waves. First we write (7.9) in the form

$$\frac{\varrho U_{11}^2}{2 v_1^2} = A + B, \tag{11.13}$$

where

$$\begin{aligned} A &\equiv f_1 + 2 v_1^2 f_2, \\ B &\equiv \sum_{\Gamma=0}^{2} v_1^{2\Gamma}\left[\frac{\partial f_\Gamma}{\partial I} + (v_2^2 + v_3^2)\frac{\partial f_\Gamma}{\partial II} + v_2^2 v_3^2 \frac{\partial f_\Gamma}{\partial III}\right]. \end{aligned} \tag{11.14}$$

Then from (11.2), (11.5) and (11.7) it follows that

$$\begin{aligned} \frac{A}{\mu} &= \frac{1}{\mu}\left[h_1 + \frac{v_1^2 - v_2^2 - v_3^2}{v_1^2 v_2^2 v_3^2} h_{-1}\right], \\ &\doteq \alpha_{10} + \alpha_{11}\vartheta + \frac{-1 - 2\vartheta + 4\delta_1}{1 + 2\vartheta}(\alpha_{-10} + \alpha_{-11}\vartheta), \\ &\doteq 1 + \tfrac{1}{2}\alpha_5\vartheta + \alpha_6\delta_1. \end{aligned} \tag{11.15}$$

Also

$$\begin{aligned} B &\doteq \frac{\partial f_0}{\partial I} + 2(1 + \vartheta - \delta_1)\frac{\partial f_0}{\partial II} + (1 + 2\vartheta - 2\delta_1)\frac{\partial f_0}{\partial III} + \\ &\quad + (1 + 2\delta_1)\left[\frac{\partial f_1}{\partial I} + 2(1 + \vartheta - \delta_1)\frac{\partial f_1}{\partial II} + (1 + 2\vartheta - 2\delta_1)\frac{\partial f_1}{\partial III}\right] + \\ &\quad + (1 + 4\delta_1)\left[\frac{\partial f_2}{\partial I} + 2(1 + \vartheta - \delta_1)\frac{\partial f_2}{\partial II} + (1 + 2\vartheta - 2\delta_1)\frac{\partial f_2}{\partial III}\right], \\ &\doteq \left(\frac{\partial}{\partial I} + 2\frac{\partial}{\partial II} + \frac{\partial}{\partial II}\right)(f_0 + f_1 + f_2) + \\ &\quad + 2(\vartheta - \delta_1)\left(\frac{\partial}{\partial II} + \frac{\partial}{\partial III}\right)(f_0 + f_1 + f_2) + \\ &\quad + 2\delta_1\left(\frac{\partial}{\partial I} + 2\frac{\partial}{\partial II} + \frac{\partial}{\partial III}\right)(f_1 + 2f_2). \end{aligned} \tag{11.16}$$

From (11.4) we see that

$$\frac{\partial}{\partial I} + 2\frac{\partial}{\partial II} + \frac{\partial}{\partial III} = \frac{1}{2}\frac{\partial}{\partial I_{\mathbf{E}}}, \qquad \frac{\partial}{\partial II} + \frac{\partial}{\partial III} = \frac{1}{4}\frac{\partial}{\partial II_{\mathbf{E}}}. \tag{11.17}$$

Hence (11.16) becomes

$$B \doteq \frac{1}{2}\frac{\partial}{\partial I_{\mathbf{E}}}(f_0 + f_1 + f_2) + \frac{1}{2}(\vartheta - \delta_1)\frac{\partial}{\partial II_{\mathbf{E}}}(f_0 + f_1 + f_2) + \delta_1\frac{\partial}{\partial I_{\mathbf{E}}}(f_1 + 2f_2). \tag{11.18}$$

Now by (11.2) and (11.4),

$$f_0 + f_1 + f_2 = h_0 + h_1 + h_{-1} - \frac{8 III_{\mathbf{E}}}{1 + 2I_{\mathbf{E}} + 4II_{\mathbf{E}} + 8III_{\mathbf{E}}} h_{-1}. \tag{11.19}$$

Hence

$$\begin{aligned} \frac{\partial}{\partial I_{\mathbf{E}}}(f_0 + f_1 + f_2) &\doteq \frac{\partial}{\partial I_{\mathbf{E}}}(h_0 + h_1 + h_{-1}), \\ &\doteq \tfrac{1}{2}\mu(\alpha_{01} + \alpha_{11} + \alpha_{-11}) + \mu(\alpha_{02} + \alpha_{12} + \alpha_{-12})\vartheta, \\ &= \mu\left(\tfrac{1}{2}\alpha_1 + \alpha_3\vartheta\right). \end{aligned} \tag{11.20}$$

Likewise

$$\begin{aligned}(\vartheta-\delta_1)\,\frac{\partial}{\partial II_{\mathbf{E}}}(f_0+f_1+f_2) &\doteq (\vartheta-\delta_1)\cdot\frac{1}{2}\mu\,(\alpha_{0\,3}+\alpha_{1\,3}+\alpha_{-1\,3}),\\ &= (\vartheta-\delta_1)\cdot\tfrac{1}{2}\mu\,\alpha_4,\end{aligned}\tag{11.21}$$

and

$$\begin{aligned}\delta_1\,\frac{\partial}{\partial I_{\mathbf{E}}}(f_1+2f_2) &\doteq \delta_1\,\frac{\partial}{\partial I_{\mathbf{E}}}(h_1-h_{-1}),\\ &\doteq \mu\,(\alpha_{1\,1}-\alpha_{-1\,1})\,\delta_1=\tfrac{1}{2}\mu\,\alpha_5\,\delta_1.\end{aligned}\tag{11.22}$$

Putting (11.20), (11.21), and (11.22) into (11.18) yields

$$\frac{B}{\mu}\doteq\frac{1}{2}\alpha_1+1+\left(\alpha_3+\frac{1}{2}\alpha_4+\frac{1}{2}\alpha_5\right)\vartheta+\left(-\frac{1}{2}\alpha_4+\frac{1}{2}\alpha_5+\alpha_6\right)\delta_1.\tag{11.23}$$

By substituting (11.15) and (11.23) into (11.13) and then using (11.11) we obtain the definitive result for the *second-order speeds of longitudinal principal waves:*

$$\frac{\varrho_0\,U_{11}^2}{\mu}\doteq\frac{\lambda}{\mu}+2+\left(\frac{\lambda}{\mu}+2+2\alpha_3+\alpha_4+\alpha_5\right)\vartheta+\left(\frac{2\lambda}{\mu}+4-\alpha_4+\alpha_5+2\alpha_6\right)\delta_1.\tag{11.24}$$

The formulae (11.12) and (11.24), giving the speeds of propagation explicitly and generally for the second-order theory, include and generalize a long series of particular results of this kind.

L. BRILLOUIN [1925, Eqs. (76), (77)] [1938, Eqs. (XI.125), (XI.126)] was the first to calculate the second-order wave speeds for a general hyperelastic material subject to uniform hydrostatic pressure. To reduce our results to forms corresponding to his, put $\delta_1=\delta_2=\delta_3\equiv e$. Then $U_{11}^2=U_{22}^2=U_{33}^2\equiv U_{\|}^2$, and $U_{12}^2=U_{21}^2=\cdots=U_{32}^2\equiv U_{\perp}^2$, and from (11.12) and (11.24) we see that[3]

$$\begin{aligned}\frac{\varrho_0}{\mu}\,\frac{dU_{\|}^2}{de}&=\frac{5\lambda}{\mu}+10+6\alpha_3+2\alpha_4+4\alpha_5+2\alpha_6,\\ \frac{\varrho_0}{\mu}\,\frac{dU_{\perp}^2}{de}&=5+\frac{3}{2}\alpha_5+\alpha_6.\end{aligned}\tag{11.25}$$

While L. BRILLOUIN [1925, §11] had calculated the second-order speeds of principal waves in a medium subject to uniaxial tension, apparently HUGHES & KELLY [1953, 2, Eqs. (12)] were the first to obtain a sufficient number of independent relations to determine the second-order elasticities from wave speeds in hyperelastic materials. Their method of linearization is not altogether clear to me, and I have not attempted to check their results against mine. TOUPIN & BERNSTEIN [1961, 1, Eqs. (5.12)] obtained formulae which are easier to assess. They considered a biaxial tension. To obtain results corresponding to theirs, put $U_{11}^2=v_{\|}^2$, $U_{12}^2=v_m^2$, $U_{13}^2=v_n^2$, $\delta_1=\delta_3=-\sigma\,\delta_2=-\sigma e$,

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[3] To get BRILLOUIN's results, which presuppose a hyperelastic material, put

$$\begin{aligned}\mu\,\alpha_3&=-\lambda+8A+24B,\\ \mu\,\alpha_4&=-16A,\\ \mu\,\alpha_5&=2(\lambda-\mu+8A),\\ \mu\,\alpha_6&=4(\mu+6C).\end{aligned}\tag{F}$$

where σ is to be held constant as $e \to 0$. Then from (11.12) and (11.24) we see that[4]

$$\begin{aligned} \frac{\varrho_0}{\mu} \frac{d v_{\|}^2}{d e} &= (1-2\sigma)\left(\frac{\lambda}{\mu}+2+2\alpha_3+\alpha_4+\alpha_5\right)-\sigma\left(\frac{2\lambda}{\mu}+4-\alpha_4+\alpha_5+2\alpha_6\right), \\ \frac{\varrho_0}{\mu} \frac{d v_m^2}{d e} &= 1+\frac{1}{2}(1-2\sigma)\,\alpha_5+\frac{1}{2}(1-\sigma)\,\alpha_6, \\ \frac{\varrho_0}{\mu} \frac{d v_n^2}{d e} &= 1+\frac{1}{2}(1-2\sigma)\,\alpha_5-\sigma\,\alpha_6. \end{aligned} \tag{11.26}$$

TOUPIN & BERNSTEIN have developed alternative forms of these results, suitable for immediate comparison with experimental data.

Let us now suppose that the 9 squared wave speeds U_{ab}^2 are known, possibly from experimental data, as linear functions of the extensions δ_c. We shall find necessary and sufficient conditions that these linear expressions be compatible with second-order elasticity for some value of the coefficients α_Γ. First, the general conditions (9.1) and (9.2) for the transverse waves may be replaced by the following second-order approximations:

$$\begin{gathered} \frac{U_{12}^2-U_{21}^2}{\delta_1-\delta_2} = \frac{U_{23}^2-U_{32}^2}{\delta_2-\delta_3} = \frac{U_{31}^2-U_{13}^2}{\delta_1-\delta_3} = 2\,U_{\perp}^2, \\ U_{23}^2 = 2(\delta_2-\delta_1)\,U_{\perp}^2+\frac{1}{2}\left(U_{12}^2+U_{13}^2\right)+\frac{1}{2}\,\frac{U_{12}^2-U_{13}^2}{\delta_2-\delta_3}\,(\delta_2+\delta_3-2\delta_1), \end{gathered} \tag{11.27}$$

where $U_{\perp}^2$ is the squared speed of transverse waves according to the linearized theory. Alternatively, these conditions may easily be derived directly from (11.12). When they are satisfied, unique values for α_5 and α_6 are determined from the squared speeds of the two kinds of transverse waves corresponding to any one principal axis. *E.g.*,

$$\begin{aligned} \frac{1}{2}\alpha_6 &= \frac{U_{12}^2-U_{13}^2}{(\delta_2-\delta_3)\,U_{\perp}^2}, \\ \frac{1}{2}\alpha_5 &= \frac{1}{\vartheta}\left[-2\delta_1-\frac{1}{4}(2\vartheta-\delta_3)\,\alpha_6+\frac{U_{12}^2+U_{13}^2}{2\,U_{\perp}^2}-1\right]-1. \end{aligned} \tag{11.28}$$

For the longitudinal waves it is easier to procede directly from (11.24). In this way one sees that the three differential conditions (9.7) are equivalent in second-order approximation to a single algebraic one:

$$\left(U_{11}^2-U_{\|}^2\right)(\delta_2-\delta_3)+\left(U_{22}^2-U_{\|}^2\right)(\delta_3-\delta_1)+\left(U_{33}^2-U_{\|}^2\right)(\delta_1-\delta_2)=0, \tag{11.29}$$

where $U_{\|}^2$ is the squared speed of longitudinal waves according to the linearized theory. If this condition is satisfied, and if α_5 and α_6 are previously specified by (11.28) or otherwise, unique values for α_3 and α_4 are determined from the squared speeds of any two kinds of longitudinal waves. *E.g.*,

$$\begin{aligned} \alpha_3 &= \frac{\left(U_{11}^2-U_{\|}^2\right)\delta_2-\left(U_{22}^2-U_{\|}^2\right)\delta_1}{2\nu(\delta_2-\delta_1)\,U_{\perp}^2}+\frac{U_{22}^2-U_{11}^2}{2(\delta_2-\delta_1)\,U_{\perp}^2}-\frac{3}{2}\left(2+\frac{\lambda}{\mu}\right)-\alpha_5-\alpha_6, \\ \alpha_4 &= -\frac{U_{22}^2-U_{11}^2}{(\delta_2-\delta_1)\,U_{\perp}^2}+2\left(2+\frac{\lambda}{\mu}\right)+\alpha_5+\alpha_6. \end{aligned} \tag{11.30}$$

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[4] To get TOUPIN & BERNSTEIN'S results, which presuppose a hyperelastic material, put

$$\begin{aligned} \mu\,\alpha_3 &= -\lambda+\nu_2+\tfrac{1}{2}\nu_1, \\ \mu\,\alpha_4 &= -2\nu_2, \\ \mu\,\alpha_5 &= 2(\lambda-\mu+\nu_2), \\ \mu\,\alpha_6 &= 4(\mu+\nu_3). \end{aligned} \tag{G}$$

In any theory in which the squared speeds emerge as linear functions of the extensions, no satisfactory conditions for the speeds to be real can result, for no linear function is positive-definite. Since the speeds cannot be non-constant linear functions of the extensions according to any exact stress-strain relation, this result does not indicate any defect of the general theory. The second-order theory is only an approximation; the resulting formulae are expected to be valid only for small extensions; and for sufficiently small extensions, the second-order wave speeds are always real if the first-order ones are. Thus no restriction on the coefficients can be found by considering the second-order wave speeds. Restrictions may result by applying the various conditions of §10 to the third-order theory of elasticity, but I will not enter this forbidding avenue of research.

The formulae and analysis of this section when compared to those preceding furnish an example of the frequent observation that an exact and general theory is often simpler and easier to understand than an approximate one or the result of a perturbation process. It is possible, of course, but less enlightening to calculate the second-order wave speeds directly from the second-order stress-strain relations, given as Eq. (E) in the footnote on p. 283.

12. Hyperelastic materials

When there exists a strain energy $\Sigma(x^m{}_{,\beta}, \boldsymbol{d}_\Lambda)$, the function $\mathfrak{h}_k^\alpha$ in (2.3) assumes the form [CFT, Eq. (303.5)]

$$\mathfrak{h}_k^\alpha = \frac{\partial \Sigma}{\partial x^k{}_{,\alpha}}. \tag{12.1}$$

From (2.5) we see that then

$$A_{km}{}^{\alpha\beta} = A_{mk}{}^{\beta\alpha}. \tag{12.2}$$

Now by (2.8),

$$Q_{km}(\boldsymbol{n}) - Q_{mk}(\boldsymbol{n}) = [A_{km}{}^{(\alpha\beta)} - A_{mk}{}^{(\alpha\beta)}]\, x^p{}_{,\alpha}\, x^q{}_{,\beta}\, n_p\, n_q. \tag{12.3}$$

Hence (12.2) implies that $Q_{km} = Q_{mk}$; thus[1] *the acoustical tensor of a hyperelastic material is symmetric; consequently all acoustic axes are real, and for each direction of propagation there exists at least one mutually orthogonal set of acoustic axes*[2].

In the hyperelastic case, accordingly, much more specific results hold. When we apply the principle of material indifference, we may express all results in

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[1] The condition is sufficient, not necessary. In order that $Q_{km}(\boldsymbol{n}) = Q_{mk}(\boldsymbol{n})$ for every $\boldsymbol{n}$, it is necessary that

$$A_{km}{}^{\alpha\beta} - A_{mk}{}^{\beta\alpha} = L_{km}{}^{\alpha\beta}, \tag{H}$$

where

$$L_{km}{}^{\alpha\beta} = L_{mk}{}^{\alpha\beta} = -L_{km}{}^{\beta\alpha}. \tag{I}$$

In view of the theorem of BERNSTEIN given below, in order to construct a non-trivial example satisfying Eqs. (H) and (I) it is necessary to consider materials violating the principle of material indifference. Since no linear constitutive equation can satisfy that principle, take $\mathfrak{h}_k^\alpha = K_k{}^\alpha{}_m{}^\beta\, x^m{}_{,\beta}$, where $K_k{}^\alpha{}_m{}^\beta = \text{const}$. Then $A_{km}{}^{\alpha\beta} = K_k{}^\alpha{}_m{}^\beta$, and solutions of Eqs. (H) and (I) are many.

[2] This theorem is due to HADAMARD [1901] [1903, 1, p. 268], generalizing earlier work of CHRISTOFFEL and HUGONIOT.

HADAMARD's theorem was questioned by DUHEM [1903] [1906. Ch. I, §§ II, IV]; while I have not attempted to follow the long analysis of DUHEM, I suspect that only some misunderstanding of notation is involved.

terms of the tensors B^{kmpq} and C^{kmpq} of §6. From (6.7) and (12.2), then,

$$B^{kmpq}=B^{pqkm}; \tag{12.4}$$

consequently, by (6.9),

$$C^{kmpq}=C^{pqkm}. \tag{12.5}$$

Eqs. (12.2), (12.4), and (12.5) are alternative necessary and sufficient condition that an elastic material be hyperelastic.

Bernstein[2a] has pointed out that when the principle of material indifferenc is adopted, a definite and simple converse to the foregoing theorem comes int force: *If for every direction of propagation the acoustical tensor of an elastic material is symmetric, then the material is hyperelastic.* For by (6.11) we see that $Q^{km}(\boldsymbol{n})=$ $Q^{mk}(\boldsymbol{n})$ if and only if

$$(B^{kpmq}-B^{mpkq})\,n_p\,n_q=0. \tag{12.5 A}$$

For this condition to hold for every $\boldsymbol{n}$, it is necessary and sufficient that

$$B^{kpmq}-B^{mpkq}+B^{kqmp}-B^{mqkp}=0. \tag{12.5 B}$$

Hence

$$B^{pkqm}-B^{qkpm}+B^{pmqk}-B^{qmpk}=0. \tag{12.5 C}$$

Adding these two equations yields

$$B^{kpmq}+B^{pkqm}-B^{mqkp}-B^{qmpk}=B^{mpkq}-B^{pmqk}+B^{qkpm}-B^{kqmp}. \tag{12.5 D}$$

By four uses of the identity $(6.10)_3$ we see that the right-hand side vanishes, while the left-hand side reduces to $2(B^{kpmq}-B^{kqmp})$. Hence (12.4) holds, showing that the material is hyperelastic.

It would be useful to express the theorem and its converse directly in terms of acoustical data, but this does not seem to be possible. Rephrasing a theorem of Kelvin & Tait on matrices [CFT, App. §37], we see that if the squared speeds are non-negative and if an orthogonal triad of acoustical axes exists, then the acoustical tensor is symmetric. From this fact and from Bernstein's theorem we may conclude that *if, in an elastic material, there exist three orthogonal acoustical axes for every direction of propagation, and if the corresponding speeds of propagation are real or purely imaginary, the material is hyperelastic.* This result fails to be a true physical converse to Hadamard's theorem because purely imaginary speeds of propagation do not seem to be distinguishable from complex speeds, since neither correspond to real singular surfaces, while both lead to amplified infinitesimal oscillations. However, it shows that the existence of orthogonal axes with real speeds of propagation for every direction is *sufficient* for an elastic material to be hyperelastic. Of course it is possible that in an elastic material which is not hyperelastic, the acoustical tensor may be symmetric for some particular directions of propagation. For example, it follows at once from (7.8) that the acoustical tensor for any principal direction of stress in an isotropic material is always symmetric, whether or not there be a stored energy.

[*Note added in proof, September 9, 1961.* The case last mentioned in small type is truly exceptional, for I have just discovered that *if in an isotropic elastic material the acoustical tensor $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric for one single direction $\boldsymbol{n}$ not lying in the plane of two principal directions, then the material is hyperelastic.* To prove this result, we begin by writing out the components of $T^{km}{}_{pq}$ in a principal coordinate system. Using (7.12), we see from (7.5) that

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[2a] Letters dated July 28 and August 8, 1961. The paragraph beginning here and the following text in small type were added in proof, September 7, 1961.

$$\begin{gathered} T^{11}{}_{11} = \frac{\partial t_1}{\partial v_1^2}, \qquad T^{11}{}_{22} = \frac{\partial t_1}{\partial v_2^2}, \dots \\ T^{12}{}_{12} = \frac{t_1 - t_2}{2\,(v_1^2 - v_2^2)}, \dots, \end{gathered} \tag{12.5 E}$$

while the other types of components of $T^{km}{}_{pq}$ vanish. Let the components of $\boldsymbol{n}$ be $\cos\vartheta_1$, $\cos\vartheta_2$, $\cos\vartheta_3$; then from (7.7) we find that

$$Q^1_2(\boldsymbol{n}) = \frac{\tilde{\varrho}}{\varrho}\left[\frac{\partial t_1}{\partial \log v_2} + \frac{t_1 - t_2}{v_1^2 - v_2^2}\, v_1^2\right] \cos\vartheta_1 \cos\vartheta_2, \tag{12.5 F}$$

while Q^2_1 is obtained by interchanging 1 and 2 in this formula. For a principal wave, $\cos\vartheta_1 \cos\vartheta_2 = 0$, so that $Q^1_2 = Q^2_1 = 0$, as we have shown in §7 by a different argument. If, however, $\cos\vartheta_1 \cos\vartheta_2 \neq 0$, then $Q^1_2 = Q^2_1$ if and only if

$$\frac{\partial t_1}{\partial \log v_1} - \frac{\partial t_1}{\partial \log v_2} = t_2 - t_1. \tag{12.5 G}$$

In an earlier work [1952, Eq. $(41.11)_2$] I have shown that this equation and the similar ones obtained from it by permutations of indices are necessary and sufficient conditions for the existence of a stored energy. Q.E.D.

If $\cos\vartheta_3 = 0$ but $\cos\vartheta_1 \cos\vartheta_2 \neq 0$, so that the wave is not a principal wave, but $\boldsymbol{n}$ lies in the plane of the principal directions corresponding to the principal stretches v_1 and v_2, then $Q^2_3 = Q^3_2 = Q^3_1 = Q^1_3 = 0$, but the single condition (12.5 G) is necessary and sufficient that $Q^1_2 = Q^2_1$. It is thus possible for special isotropic elastic but not hyperelastic materials to have symmetric acoustical tensors for all directions of propagation lying in one and only one of the three planes defined by the pairs of principal axes.

Therefore, there are three possible kinds of isotropic materials as far as acoustical properties are concerned:

1. Those for which $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric for the principal waves only.
2. Those for which $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric for the principal waves and also for any $\boldsymbol{n}$ perpendicular to one and only one of the principal axes.
3. Those for which $\boldsymbol{Q}(\boldsymbol{n})$ is symmetric for all $\boldsymbol{n}$, these last constituting the hyperelastic materials.]

Three further definite advances over the results of the more general theory may now be made. First, if we set

$$P^{pq} \equiv B_k^{pkq}, \tag{12.6}$$

from (12.4) it follows that $\boldsymbol{P}$ is symmetric. Also, if for a unit vector $\boldsymbol{n}$ we set

$$m(\boldsymbol{n}) \equiv P^{pq}\, n_p\, n_q, \tag{12.7}$$

from (6.11) and (2.9) we see that

$$m(\boldsymbol{n}) = \frac{\varrho}{\tilde{\varrho}}\, Q^k_k(\boldsymbol{n}) = \varrho \sum_{a=1}^{3} U_a^2, \tag{12.8}$$

where $\tilde{\varrho}\, U_a^2$ is the a^{th} proper number of $\boldsymbol{Q}(\boldsymbol{n})$. Since $\boldsymbol{P}$ is real and symmetric, there are at least three mutually orthogonal directions $\boldsymbol{n}$ in which $m(\boldsymbol{n})$, as given by (12.7), is extreme. By (12.8), then, *in a hyperelastic material there exist at each point at least three mutually orthogonal directions* $\boldsymbol{n}_a$ *such that the sum of the squares of the speeds of propagation of the possible waves travelling in the* *direction* $\boldsymbol{n}_a$ *is an extremum*[3]. In the foregoing statement, the possibility tha some wave speeds are purely imaginary is not excluded.

Second, as is clear from (9.2), *the S-E condition is necessary and sufficien that all squared wave speeds be positive.* In particular, if the *S-E* condition i satisfied, there can be no material singular surfaces.

Third, *the C-N condition ensures that at least one wave may travel in the direction of* $\boldsymbol{n}$, *for any* $\boldsymbol{n}$. This result was discovered by NOLL[4] and was proved a follows by ERICKSEN[5]. The proper numbers of $\boldsymbol{Q}(\boldsymbol{n})$ are the extrema of th bounded continuous function

$$F(\boldsymbol{l}) \equiv Q^{km}(\boldsymbol{n})\, l_k\, l_m, \tag{12.9}$$

where $\boldsymbol{l}$ is a unit vector. By (10.5), $F(\boldsymbol{n}) > 0$. Hence at least one extreme of $F(\boldsymbol{l})$ must be positive; that is, $\boldsymbol{Q}(\boldsymbol{n})$ has at least one positive proper number, so that the corresponding wave speed is real.

For isotropic materials it is easy to show[6] that the response coefficients f_Γ in (7.1) have the following special forms:

$$\frac{1}{2} f_0 = \sqrt{III}\,\frac{\partial\Sigma}{\partial III}, \qquad \frac{1}{2}\frac{\varrho_0}{\varrho} f_1 = \frac{\partial\Sigma}{\partial I} + I\,\frac{\partial\Sigma}{\partial II}, \qquad \frac{1}{2}\frac{\varrho_0}{\varrho} f_2 = -\frac{\partial\Sigma}{\partial II}, \tag{12.10}$$

where $\Sigma(I, II, III)$ is the strain energy. Hence

$$2\frac{\partial\Sigma}{\partial I} = \frac{\tilde\varrho}{\varrho}(f_1 + I f_2), \qquad 2\frac{\partial\Sigma}{\partial II} = -\frac{\tilde\varrho}{\varrho} f_2, \qquad 2\frac{\partial\Sigma}{\partial III} = \frac{1}{\sqrt{III}} f_0. \tag{12.11}$$

Accordingly, conditions of integrability for Σ are

$$\begin{aligned}
\frac{\partial}{\partial I}\left(\frac{f_2}{\varrho}\right) + \frac{\partial}{\partial II}\left(\frac{f_1}{\varrho} + I\,\frac{f_2}{\varrho}\right) &= 0,\\
\frac{1}{\tilde\varrho\sqrt{III}}\,\frac{\partial f_0}{\partial I} - \frac{\partial}{\partial III}\left(\frac{f_1}{\varrho} + I\,\frac{f_2}{\varrho}\right) &= 0,\\
\frac{1}{\tilde\varrho\sqrt{III}}\,\frac{\partial f_0}{\partial II} + \frac{\partial}{\partial III}\left(\frac{f_2}{\varrho}\right) &= 0.
\end{aligned} \tag{12.12}$$

By use of (9.3) and (9.6) we may express these conditions directly in terms of the wave speeds:

$$\begin{aligned}
\frac{\partial}{\partial I}\left[\frac{U_{12}^2 - U_{13}^2}{v_1^2(v_2^2 - v_3^2)}\right] + \frac{\partial}{\partial II}\left[\frac{v_2^2 U_{12}^2 - v_3^2 U_{13}^2}{v_1^2(v_2^2 - v_3^2)}\right] &= 0,\\
\frac{1}{\tilde\varrho\sqrt{III}}\,B_1 - \frac{\partial}{\partial III}\left[\frac{v_2^2 U_{12}^2 - v_3^2 U_{13}^2}{v_1^2(v_2^2 - v_3^2)}\right] &= 0,\\
\frac{1}{\tilde\varrho\sqrt{III}}\,B_2 + \frac{\partial}{\partial III}\left[\frac{U_{12}^2 - U_{13}^2}{v_1^2(v_2^2 - v_3^2)}\right] &= 0.
\end{aligned} \tag{12.13}$$

We presume that all the conditions of compatibility obtained in §9 are satisfied; then, in particular, B_1 and B_2 and the two expressions in brackets in (11.13) are symmetric functions of the v_a and hence are equal to functions of I, II, III. By use of the conditions of compatibility, moreover, A_1, A_2, and A_3, as defined by (9.4), are expressible in terms of U_{12}^2, U_{13}^2, U_{11}^2, U_{22}^2, U_{33}^2, their first derivatives, and v_1, v_2, v_3; thus B_1 and B_2, as defined by $(9.6)_{1,2}$, are functions of these same quantities. The results obtained may be summarized as follows: *for an isotropic elastic material, let the three possible squared wave-speeds corresponding to one principal axis of stress and strain as well as the remaining two longitudinal wave-speeds be given as functions of the principal stretches; then the material is hyperelastic if and only if the three differential conditions* (12.13) *are satisfied.*

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[3] This result generalizes slightly a theorem of PASTORI [1949, §6]. It is obvious that $\varrho\sum_{a=1}^{3} U_a^2(\boldsymbol{n})$, being a bounded continuous real function of $\boldsymbol{n}$, has a maximum and a minimum unless it is constant. It is the existence of the third extremum that does not seem evident *a priori*.

[4] Letter dated September 13, 1960.

[5] Letter dated June 16, 1961. In a letter of July 14, 1961, NOLL informed me that this proof is the same as his and disclosed a sharper inequality for $F(\boldsymbol{l})$.

[6] The results used here are equivalent to those given by TRUESDELL [1952, Eq. (41.5)] and may be derived by a similar procedure.

The conditions (12.13), while expressed directly in terms of the speeds of principal waves, are elaborate in form and difficult to interpret. BERNSTEIN'S condition, namely, that $\boldsymbol{Q}(\boldsymbol{n})$ shall be symmetric for every $\boldsymbol{n}$, is simple but does not seem to lend itself to interpretation in terms of acoustical data alone. It assures us that an elastic material in which three waves of mutually orthogonal amplitudes may propagate in any direction must have a stored energy, but this is not a characterizing condition. In respect to propagation of waves, there seems to be no especially simple property by which the existence or non-existence of a stored energy manifests itself without fail.

In the second-order theory, however, a simple identity, suitable for experimental test, replaces the elaborate general conditions (12.13). Either from (12.13), or from the formulae (E), (F), or (G) given in the footnotes on pp. 283, 285, and 286, respectively, it is easy to see that for the existence of a stored energy in the second-order theory of isotropic materials it is necessary and sufficient that the 6 first-order and second-order elasticities be reduced to 5 by the following equation:

$$\alpha_4 + \alpha_5 = 2\left(\frac{\lambda}{\mu} - 1\right). \tag{12.14}$$

This condition may be expressed directly in terms of the wave speeds by means of (11.28) and (11.29), but the resulting identity is complicated. An equivalent but simpler formula may be obtained by differentiation of (11.12) and (11.24). Since

$$\begin{aligned} \frac{\varrho_0}{\mu}\,\frac{\partial U_{12}^2}{\partial \delta_1} &= 3 + \frac{1}{2}\alpha_5 + \frac{1}{2}\alpha_6, \\ \frac{\varrho_0}{\mu}\left(\frac{\partial}{\partial \delta_1} - \frac{\partial}{\partial \delta_2}\right) U_{11}^2 &= \frac{2\lambda}{\mu} + 4 - \alpha_4 + \alpha_5 + 2\alpha_6, \end{aligned} \tag{12.15}$$

we see that (12.14) is equivalent to the following ***universal criterion for a stored energy*** in the second-order theory:

$$\left(\frac{\partial}{\partial \delta_1} - \frac{\partial}{\partial \delta_2}\right) U_{11}^2 - 4\,\frac{\partial U_{12}^2}{\partial \delta_1} + 6\,U_{\perp}^2 = 0. \tag{12.16}$$

This condition affords the first opportunity, so far as I know, for a clear-cut and unequivocal experimental test of the existence of a stored energy in an isotropic material[7]. It is different from the previously derived conditions of compatibility in that *it connects a longitudinal-wave speed with a transverse-wave speed*. There are in fact 6 conditions of this kind, but, on the presumption that the general identities (11.27) and (11.29) are satisfied, any of them implies the rest. In this sense (12.16) is *necessary and sufficient* for the existence of a second-order stored energy. If the partial derivatives are evaluated in the natura state, (12.16) is also a necessary but not sufficient condition for the existence of a stored energy according to the general theory.

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[7] In the linearized theory, every isotropic elastic material is hyperelastic. In the finite theory, this is not so; while practically all previous researches on the finite theory concern hyperelastic materials, I know of no case in which the existence of a stored energy is necessary for the results obtained. In fact, many of the solutions derived by the British school could be expressed more simply in terms of the coefficients h_Γ, independently of whether there be a stored energy.

In order to test (12.16) properly it is necessary to let δ_1, δ_2 and δ_3 vary *independently*. It is impossible, for example, to find a relation corresponding to (12.16) by considering a one-parameter family of deformations such as that used by Toupin & Bernstein and explained in §11. It may be preferable, however, to reduce the number of independent variables from three to two. For example, if we employ only deformations such that $\delta_2 = \delta_3$, then it is easy to show that instead of (12.16) we have the alternative necessary and sufficient condition

$$\left(\frac{\partial}{\partial\delta_1} - \frac{1}{2}\frac{\partial}{\partial\delta_2}\right) U_{11}^{2} - 4\frac{\partial U_{12}^{2}}{\partial\delta_1} + 6U_{\perp}^{2} = 0, \tag{12.17}$$

but δ_1 and δ_2 must be allowed to vary independently.

13. Duhem's theorems on the effect of temperature variation

In general the stress-strain relation (2.3) involves an additional scalar parameter such as the temperature, ϑ, or the specific entropy, η:

$$T_k^\alpha = \hat{\mathfrak{h}}_k^\alpha(x^m{}_{,\beta}, \boldsymbol{d}_\Lambda, \eta) = \bar{\mathfrak{h}}_k^\alpha(x^m{}_{,\beta}, \boldsymbol{d}_\Lambda, \vartheta), \tag{13.1}$$

where $\hat{\mathfrak{h}}$ and $\bar{\mathfrak{h}}$ are different functions. The results obtained so far are valid when $\eta = \text{const.}$ or when $\vartheta = \text{const.}$, but the tensors $\boldsymbol{A}$, defined by (2.5), and the acoustical tensors, $\boldsymbol{Q}$, defined by (2.7), are different in the two cases and may be distinguished by the notations $\hat{\boldsymbol{A}}$, $\bar{\boldsymbol{A}}$, $\hat{\boldsymbol{Q}}$, $\bar{\boldsymbol{Q}}$.

As far as I can learn, the only treatment of more general conditions, when temperature and entropy may vary within the body, is that given by Duhem [1903, 2] [1906, Ch. I, §2]. I have despaired at following his long calculations, made more difficult from an unfortunate choice of variables, but here I present my own treatment of the same problems, leading to results which seem to be equivalent to his.

First, if we substitute (13.1) into (2.4), we obtain equations of motion of the following forms:

$$\begin{aligned} \hat{A}_{km}{}^{(\alpha\beta)}\, x^m{}_{;\alpha\beta} + \frac{\partial\hat{\mathfrak{h}}_k^\alpha}{\partial d_\Lambda^m} d_{\Lambda;\alpha}^m + \frac{\partial\hat{\mathfrak{h}}_k^\alpha}{\partial\eta}\eta_{,m}\, x^m{}_{,\alpha} + \tilde{\varrho} f_k &= \tilde{\varrho}\,\ddot{x}_k, \\ \bar{A}_{km}{}^{(\alpha\beta)}\, x^m{}_{;\alpha\beta} + \frac{\partial\bar{\mathfrak{h}}_k^\alpha}{\partial d_\Lambda^m} d_{\Lambda;\alpha}^m + \frac{\partial\mathfrak{h}_k^\alpha}{\partial\vartheta}\vartheta_{,m}\, x^m{}_{,\alpha} + \tilde{\varrho} f_k &= \tilde{\varrho}\,\ddot{x}_k. \end{aligned} \tag{13.2}$$

Taking the jump of these equations at a singular surface leads to propagation conditions of the form (2.7) not only if $\eta = \text{const.}$ or if $\vartheta = \text{const.}$ but also under the weaker assumptions

$$[\eta_{,m}] = 0, \qquad [\vartheta_{,m}] = 0, \tag{13.3}$$

respectively. We now consider physical circumstances such as to render valid one or the other of the relations (13.3).

First, we review a classical argument, given in effect by FOURIER [*cf.* CFT, §296], which shows that if $[\vartheta]=0$ and if FOURIER's law of heat conduction is assumed, then $(13.3)_2$ follows. For by MAXWELL's theorem [CFT, Eq. (175.8)], the condition $[\vartheta]=0$ implies that $[\vartheta_{,k}]=B\,n_k$. FOURIER's law of heat conduction asserts that the flux of energy, $\boldsymbol{h}$, depends linearly on $\vartheta_{,k}$, that is, $h_m=K_m^{\ p}\,\vartheta_{,p}$, where $\boldsymbol{K}$ is positive-definite. Hence $[h_m]=B K_m^{\ p}\,n_p$, where $\boldsymbol{K}$ is assumed continuous, and consequently $\boldsymbol{n}\cdot[\boldsymbol{h}]=B K_m^{\ p}\,n_p\,n^m$. But the principle of conservation of energy requires that $\boldsymbol{n}\cdot[\boldsymbol{h}]=0$, provided the internal energy be continuous. Hence $B=0$, whence $(13.3)_2$ follows.

The foregoing argument has nothing to do with elasticity, but we may summarize its implication in an elastic context by the following ***theorem on elastic conductors:*** *In an elastic material subject to* FOURIER's *law of heat conduction, the laws of propagation for waves of second order across which the temperature, internal energy, and heat conductivity are continuous are the same as those of homothermal waves.* Similarly, if we consider a wave of third order across which ε, $\vartheta_{,k}$, $K_m^{\ p}$, and $K_{m\ ,q}^{\ p}$ are assumed continuous, from the differential form of the equation of energy we conclude that $[\vartheta_{,km}]=0$, whence it follows that the laws of propagation are again those of homothermal waves. Parallel assumptions lead to a parallel theorem for waves of higher order.

Second, assume that there is no conduction of heat at all: $\boldsymbol{h}=0$. Then[1] $\dot\eta=0$, and consequently $[\dot\eta]=0$. If it is assumed that $[\eta]=0$, then by HADAMARD's conditions [CFT, Eq. (180.5)] we infer that $(13.3)_1$ holds. Thus we have the following ***theorem on elastic non-conductors***[2]***:*** *in an elastic material that does not conduct heat, the laws of propagation of acceleration waves are the same as for the homentropic case.* Parallel assumptions lead to a parallel theorem for waves of higher order.

We may relate the tensors $\hat{\boldsymbol{A}}$ and $\bar{\boldsymbol{A}}$ if we assume an equation of state for the temperature:

$$\vartheta=\hat\vartheta(x^m_{\ ,\beta},\boldsymbol{d}_\Lambda,\eta)\,. \tag{13.4}$$

Then from (13.1) and (2.5) we see that

$$\hat{A}_{km}^{\ \ \ \alpha\beta}=\bar{A}_{km}^{\ \ \ \alpha\beta}+\frac{\partial\bar{\mathfrak{h}}_k^\alpha}{\partial\vartheta}\,\frac{\partial\hat\vartheta}{\partial x^m_{\ ,\beta}}\,. \tag{13.5}$$

While this result is not illuminating in general, for hyperelastic materials it can be put into a form having immediate interpretation. In the hyperelastic case there is an equation of state for the internal energy,

$$\varepsilon=\hat\varepsilon\,(x^p_{\ ,\gamma},\boldsymbol{d}_\Lambda,\eta)\,, \tag{13.6}$$

while

$$T_k^\alpha=\hat{\mathfrak{h}}_k^\alpha=\tilde\varrho\,\frac{\partial\hat\varepsilon}{\partial x^k_{\ ,\alpha}}\,,\qquad \vartheta=\hat\vartheta=\frac{\partial\hat\varepsilon}{\partial\eta} \tag{13.7}$$

[*cf.* CFT, §256A]. Then

$$\frac{\partial\hat\vartheta}{\partial x^m_{\ ,\beta}}=\frac{\partial^2\hat\varepsilon}{\partial x^m_{\ ,\beta}\,\partial\eta}=\frac{1}{\tilde\varrho}\,\frac{\partial\hat{\mathfrak{h}}_k^\alpha}{\partial\eta}\,. \tag{13.8}$$

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[1] For hyperelastic materials, this may be proved [*cf.* CFT, §256A]; for more general elastic materials, it is offered as a hypothesis.

[2] The special case of this theorem for gas dynamics is familiar: The speed of sound for isentropic flow is the same as for a homentropic flow at the same thermodynamic state.

But by substituting $(13.7)_3$ into $(13.1)_2$ and differentiating with respect to we see that

$$\frac{\partial \hat{\mathfrak{h}}_m{}^\beta}{\partial \eta} = \frac{\partial \overline{\mathfrak{h}}_m{}^\beta}{\partial \vartheta} \frac{\partial \hat{\vartheta}}{\partial \eta} = \frac{\vartheta}{\varkappa} \frac{\partial \overline{\mathfrak{h}}_m{}^\beta}{\partial \vartheta}, \qquad (13.9$$

where $\varkappa$ is the specific heat at constant deformation. Substituting (13.9) in (13.5) yields a fundamental relation between the homentropic and homotherm elasticities of any hyperelastic material:

$$\hat{A}_{km}{}^{\alpha\beta} = \overline{A}_{km}{}^{\alpha\beta} + \frac{\vartheta}{\varkappa \tilde{\varrho}} \frac{\partial \overline{\mathfrak{h}}_k{}^\alpha}{\partial \vartheta} \frac{\partial \overline{\mathfrak{h}}_m{}^\beta}{\partial \vartheta}. \qquad (13.10$$

Using (2.8), we derive DUHEM'S ***fundamental relation*** between the two acoustical tensors:

$$\hat{Q}_{km} = \overline{Q}_{km} + \frac{\vartheta}{\varkappa \tilde{\varrho}} P_k P_m \qquad (13.11$$

where

$$P_k \equiv \frac{\partial \overline{\mathfrak{h}}_k{}^\alpha}{\partial \vartheta} x^p{}_{,\alpha} n_p. \qquad (13.12$$

Thus the properties of homentropic waves may be calculated directly from the stress-strain relations in which temperature is the thermodynamic parameter Moreover, the tensor $P_k P_m$ is always non-negative definite, for any $\boldsymbol{n}$. Therefore, since $\vartheta/\varkappa \geqq 0$, it follows that if $\overline{\boldsymbol{Q}}(\boldsymbol{n})$ is positive-definite, so is $\hat{\boldsymbol{Q}}(\boldsymbol{n})$. Thus as remarked by DUHEM, *if, for a given hyperelastic material subject to given deformation, three independent waves may propagate in the direction* $\boldsymbol{n}$ *if* $\vartheta =$ const. *then the same holds if* $\eta =$ const. In other words, if the homothermal wave speed are real, so are the homentropic wave speeds, but the converse need not hold

14. Remarks on other researches

In this paper I have not attempted to summarize everything known regardin waves in severely deformed elastic materials[1], but some topics should be liste for reference.

1. The connection between HADAMARD'S stability and wave propagation already mentioned.

2. JOUGUET'S presentation of the general shock conditions in an elastic contex and analysis of weak shocks [1920, 1—4] [1921].

3. L. BRILLOUIN'S elaborate study of radiation pressure and other second order effects [1925] [1938, Ch. XI, §§ VII—XVI].

4. ERICKSEN'S analysis [1953, 1] of waves in isotropic incompressible hyperelastic materials. By combining the approach of ERICKSEN with that presente in §§ 2—6 of this paper, a complete theory of waves in general incompressibl elastic materials is easily constructed. Such waves are always transverse [CFT §191]. For isotropic materials the wave speeds are given by (7.10); the relation (7.15), the conditions of compatibility (9.1) and (9.2), and the determination of the stress-strain relations from the wave speeds by means of (9.3) remain valid but may be simplified a little since $v_1^2 v_2^2 v_3^2 = 1$. The second-order theory for a general isotropic incompressible material may be read off from ERICKSEN'S

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[1] Such a summary will appear in the article by W. NOLL & C. TRUESDELL, *Th Non-linear Field Theories of Mechanics*, FLÜGGE'S Handbuch der Physik, Vol. 8 part III, in preparation.

exact theory for a Mooney material, since such a material serves as a general second-order approximation. In addition to the 4 conditions of compatibility (11.27), there is 1 more:

$$\frac{U_{12}^2+U_{13}^2-2U_\perp^2}{\delta_1}-\frac{U_{12}^2-U_{13}^2}{\delta_2-\delta_3}=4U_\perp^2. \tag{14.1}$$

Thus any one speed determines all possible speeds; furthermore, the single second-order elasticity is also determined by any one wave speed; in fact, it suffices to consider a one-parameter family of deformations. As is well known, every isotropic incompressible elastic material is also hyperelastic as far as the second-order theory is concerned.

5. Toupin & Bernstein's conditions of compatibility for waves according to the linearized theory of anisotropic materials [1961, 1, §4]. These are easily generalized, in slightly modified form, to the case when there need be no stored energy.

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Note added in proof, September 19, 1961. Shock relations in hyperelastic materials are derived and discussed qualitatively by J. Dewey: Strong shocks and stress-strain relations in solids. Part I. Ballistic Research Laboratories Report No. 1074, Aberdeen Proving Grounds, Maryland, April, 1959. Plane infinitesimal oscillations in a biaxially stretched isotropic hyperelastic material that conducts heat according to thermoelastic constitutive equations,

$$\boldsymbol{\Sigma}=\boldsymbol{\Sigma}(\boldsymbol{B},\vartheta), \qquad \boldsymbol{h}=\boldsymbol{K}\cdot\nabla\vartheta,$$

where $\boldsymbol{K}$ is an isotropic function of $\boldsymbol{B}$ and $\nabla\vartheta$, have been studied by J.N. Flavin & A.E. Green: J. Mech. Phys. Solids **9**, 179—190 (1961). Except in limiting cases, the contents of these papers does not overlap the research presented above.

Not included in the list of earlier works cited by Toupin & Bernstein are the following papers on special theories of waves in stressed materials: Birch, F.: The effect of pressure upon the elastic parameters of isotropic solids, according to Murnaghan's theory of finite strain. J. Appl. Phys. **9**, 279—288 (1938). — Biot, M.A.: The influence of initial stress on elastic waves. J. Appl. Phys. **11**, 522—530 (1940).

The somewhat perplexing conclusions of these authors regarding waves in bodies subject to great hydrostatic pressure are cleared and unified by the exact theory given in §8 of the foregoing paper.

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Small Deformations Superposed on Large Deformations in Materials with Fading Memory

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1. Introduction

The non-linear stress-deformation relation for materials with memory has been discussed by Green & Rivlin [*1*] and Green, Rivlin & Spencer [*2*]. In the present paper we derive the restricted form applicable to small dynamic deformations superposed on large static deformations. Although this can be done by making an appropriate linearization of the final result of the non-linear theory, we avoid some of the complexities of that theory by introducing the linearization at an earlier stage. The results obtained will apply in particular to problems of wave propagation in finitely deformed viscoelastic bodies.

It is assumed that the stress depends on the deformation history, and that the properties of the material do not change with time. The stress components are then hereditary functionals of the deformation gradients. Because rigid rotation of a deformed body has no effect on stress, other than to rotate the stress field, and since the stress-deformation relation must embody this fact, the relation is of a certain restricted form. This form was derived by Green & Rivlin [*1*] by using invariant-theoretic methods, and subsequently Noll [*3*] derived an equivalent form. Noll's derivation is simpler than that of Green & Rivlin in that it does not make use of invariant theory, but the form he derives is not suitable for the purposes of the present paper. In Section 3 we take the opportunity to present a derivation which combines some of the simplicity of Noll's method with the explicitness of Green & Rivlin's result.

When the deformation consists of an infinitesimal displacement field superimposed on a finite deformation, the stress-deformation relation can be linearized with respect to the gradients of the infinitesimal displacement field (Section 4). In a material with fading memory, if the finite deformation is held fixed long enough so that the original deforming process has no further effect, the non-linear functionals specifying the stress can be expressed in terms of non-linear functions and linear functionals (Sections 5 and 6).

The functions appearing in the linearized formulation are subject to restrictions in form if the material has any symmetries in its undeformed state (Section 7). In the case of isotropic materials (Section 8), these restrictions are such that the constitutive equation can be brought into a simpler form involving the classical strain tensor (Section 9).

2. Materials with memory and hereditary functionals

Consider a body in its undeformed state. Let the coordinates of a generic particle be X_i, in a fixed Cartesian coordinate system x. If the body is deformed as time progresses, then the particle which was originally at X_i moves to a new position x_i at time t, and the deformation may be specified by giving x_i as a function of X_p and t:

$$x_i = x_i(X_p, t). \tag{2.1}$$

The derivatives $\partial x_i/\partial X_p$, called deformation gradients, may be used as measures of the deformation in the neighborhood of the particle X_p.

Suppose that the stress at a given particle depends on the deformation gradients at that particle, not only at the instant t considered but also at all previous times. The stress components σ_{ij} at time t are then functionals of the deformation gradients at the particle considered. To express this idea, we write

$$\sigma_{ij}(t) = f_{ij}\Big[\underset{\tau=0}{\overset{\infty}{d\, x_p(t-\tau)/\partial X_q}};\, t\Big]. \tag{2.2}$$

Explicit dependence of the functionals on t indicates that the intrinsic properties of the material may change as time progresses, whether the material is subjected to a deformation or not. This possibility will immediately be rejected, however, for we assume that the material has the following property: if two deformation histories differ only by a time shift, then the corresponding stress histories differ only by that same time shift. It is easy to see, and correspondingly easy to show rigorously, that this assumption implies that the functionals cannot depend on t explicitly. Thus in place of Eq. (2.2) we write

$$\sigma_{ij}(t) = f_{ij}\Big[\underset{\tau=0}{\overset{\infty}{\partial x_p(t-\tau)/\partial X_q}}\Big]. \tag{2.3}$$

The functionals are then said to be *hereditary*.

3. Rotation of the physical system

Not every relation of the form (2.3) is physically meaningful. When an arbitrary rigid rotation is superimposed on a given deformation history, the stress field undergoes an equal rigid rotation. The form of the constitutive equation must be such that this condition is satisfied.

Let $\bar{x}_i$ be a deformation history obtained by superimposing a rigid rotation on the deformation $x_i(t)$, so that

$$\bar{x}_i(X_p, t) = a_{ij}(t)\, x_j(X_p, t). \tag{3.1}$$

Here the coefficients $a_{ij}(t)$ must satisfy the orthogonality conditions

$$a_{ik}(t)\, a_{jk}(t) = a_{ki}(t)\, a_{kj}(t) = \delta_{ij}, \qquad |a_{ij}(t)| = 1. \tag{3.2}$$

From Eq. (3.1), the deformation gradients for the new deformation are

$$\partial \bar{x}_i(t)/\partial X_p = a_{ij}(t)\, \partial x_j(t)/\partial X_p. \tag{3.3}$$

Letting $\bar{\sigma}_{ij}(t)$ denote the stress components associated with the deformation $\bar{x}_i(t)$, Eq. (2.3) implies that

$$\bar{\sigma}_{ij}(t) = f_{ij}\Big[\underset{\tau=0}{\overset{\infty}{\partial \bar{x}_p(t-\tau)/\partial X_q}}\Big]. \tag{3.4}$$

The condition to be satisfied is that the components $\bar{\sigma}_{ij}(t)$ are related to the components $\sigma_{ij}(t)$ by the rotation $a_{ij}(t)$. Thus,

$$\bar{\sigma}_{ij}(t) = a_{ik}(t)\, a_{jl}(t)\, \sigma_{kl}(t). \tag{3.5}$$

By using Eqs. (2.3) and (3.4) in Eq. (3.5), and making use of Eqs. (3.2) and (3.3), we obtain

$$f_{ij}[\partial x_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q] = a_{ki}(t)\, a_{lj}(t)\, f_{kl}[a_{pr}(t-\tau) \underset{\tau=0}{\overset{\infty}{\partial x_r}}(t-\tau)/\partial X_q]. \tag{3.6}$$

The functionals f_{ij} must be of such form that Eq. (3.6) is satisfied no matter what the choice of $a_{ij}(t)$ may be, subject to the restrictions (3.2).

We may in particular let $a_{ij}(t)$ be the j-component of vector number i in an orthogonal system, formed from the three vectors $\partial x_j/\partial X_1$, $\partial x_j/\partial X_2$, and $\partial x_j/\partial X_3$ by the Schmidt orthogonalization process. In the first step of this process, $a_{1j}(t)$ is constructed by dividing $\partial x_j(t)/\partial X_1$ by its length. We thus obtain

$$a_{1j}(t) = [G_{11}(t)]^{-\frac{1}{2}}\, \partial x_j(t)/\partial X_1. \tag{3.7}$$

Here $G_{11}(t)$ is one of the set of inner products $G_{pq}(t)$, called strain components, which are defined by

$$G_{pq}(t) = \frac{\partial x_i(t)}{\partial X_p}\,\frac{\partial x_i(t)}{\partial X_q}. \tag{3.8}$$

Next, we construct the unit vector in the plane of $\partial x_j/\partial X_1$ and $\partial x_j/\partial X_2$, perpendicular to $\partial x_j/\partial X_1$ and forming a positive inner product with $\partial x_j/\partial X_2$. Using the notation $\Delta_{pq}(t)$ for the cofactor of $G_{pq}(t)$ in the determinant $G(t) = |G_{pq}(t)|$, this vector may be written

$$a_{2j}(t) = [G_{11}(t)\, \Delta_{33}(t)]^{-\frac{1}{2}}\,[-G_{12}(t)\, \partial x_j(t)/\partial X_1 + G_{11}(t)\, \partial x_j(t)/\partial X_2]. \tag{3.9}$$

Finally, $a_{3j}(t)$ is taken to be the unit vector perpendicular to $\partial x_j/\partial X_1$ and $\partial x_j/\partial X_2$ whose inner product with $\partial x_j/\partial X_3$ is positive. Thus,

$$a_{3j}(t) = [\Delta_{33}(t)\, G(t)]^{-\frac{1}{2}}\left[\Delta_{13}(t)\,\frac{\partial x_j(t)}{\partial X_1} + \Delta_{23}(t)\,\frac{\partial x_j(t)}{\partial X_2} + \Delta_{33}(t)\,\frac{\partial x_j(t)}{\partial X_3}\right]. \tag{3.10}$$

We note that $G_{11}(t)$, $\Delta_{33}(t)$, and $G(t)$ are positive in any deformation possible in a real material. Equations (3.7), (3.9) and (3.10) may be abbreviated as

$$a_{ij}(t) = C_{ik}(t)\, \partial x_j(t)/\partial X_k, \tag{3.11}$$

where the coefficients $C_{ik}(t)$ are functions of the strain components $G_{pq}(t)$.

Since the coefficients $a_{ij}(t)$ constructed in the manner described above satisfy the restrictions (3.2), Eq. (3.6) must be satisfied for this choice of $a_{ij}(t)$. Thus, if Eq. (3.11) is used in Eq. (3.6), and Eq. (3.8) is used to simplify the expression for the functional argument, we obtain

$$\begin{aligned} f_{ij}[\partial x_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q] &= C_{km}(t)\,\frac{\partial x_i(t)}{\partial X_m}\, C_{ln}(t)\,\frac{\partial x_j(t)}{\partial X_n}\, f_{kl}[C_{pr}(t-\tau) \underset{\tau=0}{\overset{\infty}{G_{rq}}}(t-\tau)] \\ &= \frac{\partial x_i(t)}{\partial X_m}\,\frac{\partial x_j(t)}{\partial X_n}\, F_{mn}[G_{pq}(\underset{\tau-0}{\overset{\infty}{t}} - \tau)] \quad \text{(say)}. \end{aligned} \tag{3.12}$$

The second member of this equality is of the form shown in the last member because the coefficients $C_{ij}(t)$ are functions of the strain components $G_{pq}(t)$.

It is thus seen that if Eq. (3.6) is satisfied, the functionals f_{ij} can be written in the form (3.12). It is easy to show that, conversely, if the functionals are of the form (3.12), then Eq. (3.6) is satisfied for all orthogonal transformations. The constitutive equation is therefore of the restricted form

$$\sigma_{ij}(t) = \frac{\partial x_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_l} F_{kl}[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}}]. \tag{3.13}$$

We note that since the stress tensor is symmetric, $F_{kl} = F_{lk}$.

4. Small deformations superimposed on large deformations

Consider a deformation obtained by superimposing a displacement field $\varepsilon u_i(X_p, t)$ on the basic deformation $x_i = x_i(X_p, t)$:

$$x_i^*(X_p, t) = x_i(X_p, t) + \varepsilon u_i(X_p, t). \tag{4.1}$$

The stress components corresponding to this deformation are, from Eq. (3.13),

$$\sigma_{ij}(t) = \frac{\partial x_i^*(t)}{\partial X_k} \frac{\partial x_j^*(t)}{\partial X_l} F_{kl}[\overset{\infty}{\underset{\tau=0}{G_{pq}^*(t-\tau)}}], \tag{4.2}$$

where the strain components $G_{pq}^*(t)$ are those obtained by replacing $x_i(t)$ by $x_i^*(t)$ in Eq. (3.8). We wish to obtain the linearized form of Eq. (4.2), valid for small enough values of ε.

By using Eq. (4.1) and neglecting terms of order ε^2, we obtain

$$\frac{\partial x_i^*(t)}{\partial X_k} \frac{\partial x_j^*(t)}{\partial X_l} = \frac{\partial x_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_l} + \varepsilon \left[\frac{\partial u_i(t)}{\partial X_k} \frac{\partial x_j(t)}{\partial X_l} + \frac{\partial x_i(t)}{\partial X_k} \frac{\partial u_j(t)}{\partial X_l} \right]. \tag{4.3}$$

The strain components $G_{pq}^*(t)$ appropriate to the new history can be found by setting $i = j$ in Eq. (4.3). Thus,

$$G_{kl}^*(t) = G_{kl}(t) + 2\varepsilon E_{kl}(t). \tag{4.4}$$

Here the functions $G_{kl}(t)$ are the strain components for the basic deformation $x_i(t)$, and the functions $E_{kl}(t)$ are defined by

$$2E_{kl}(t) = \frac{\partial u_i(t)}{\partial X_k} \frac{\partial x_i(t)}{\partial X_l} + \frac{\partial x_i(t)}{\partial X_k} \frac{\partial u_i(t)}{\partial X_l}. \tag{4.5}$$

We now assume that the functional F_{kl} is particularly dependent on the value of $G_{pq}^*(t-\tau)$ and its time derivatives at $\tau = 0$, and is a continuous functional of $G_{pq}^*(t-\tau)$ over the range $0 < \tau < \infty$. Using the notation

$$G_{pq}^{*(0)} = G_{pq}^*(t), \qquad G_{pq}^{*(\mu)} = d^\mu G_{pq}^*(t)/dt^\mu, \tag{4.6}$$

we suppose that F_{kl} may be expressed as the sum of a number of terms, each of which is the product of a differentiable function f_ν, say, of $G_{pq}^{*(\mu)}$ $(\mu = 0, 1, 2, \ldots)$, and a continuous functional Φ_ν, say, of $G_{pq}^*(t-\tau)$ over the range $0 < \tau < \infty$. Then,

$$F_{kl}[\overset{\infty}{\underset{\tau=0}{G_{pq}^*(t-\tau)}}] = \sum_\nu f_\nu(G_{pq}^{*(\mu)}) \, \Phi_\nu[\overset{\infty}{\underset{\tau=0}{G_{pq}^*(t-\tau)}}]. \tag{4.7}$$

Using Taylor's theorem, we have, with (4.4) and the neglect of terms of order higher than the first in ε,

$$f_\nu(G_{pq}^{*(\mu)}) = f_\nu(G_{pq}^{(\mu)}) + 2\varepsilon \sum_\mu E_{pq}^{(\mu)} \, \partial f / \partial G_{pq}^{*(\mu)} \big|_{\tau=0}. \tag{4.8}$$

In (4.8) we use the notation

$$E^{(0)}_{pq}=E_{pq}(t),\qquad E^{(\mu)}_{pq}=d^{\mu}E_{pq}(t)/dt^{\mu}. \tag{4.9}$$

By using the integral representation for continuous functionals, it is seen that neglecting terms of order ε^2, the functional Φ_ν may be expressed in the form

$$\Phi_\nu\Big[\overset{\infty}{\underset{\tau=0}{G^*_{pq}(t-\tau)}}\Big]=\Phi_\nu\Big[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}}\Big]+\varepsilon\int_0^\infty\Phi^{(\nu)}_{rs}\Big[\overset{\infty}{\underset{\xi=0}{G_{pq}(t-\xi)}};\tau\Big]E_{rs}(t-\tau)\,d\tau. \tag{4.10}$$

By using (4.8) and (4.10) in (4.7), and again neglecting terms of order ε^2, we see that F_{kl} is expressible in the form

$$\begin{aligned}F_{kl}\Big[\overset{\infty}{\underset{\tau=0}{G^*_{pq}(t-\tau)}}\Big]&=F_{kl}\Big[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}}\Big]+\\&\quad+\varepsilon\sum_{\nu}A^{(\nu)}_{klrs}\Big[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}};G^{(\mu)}_{pq}\Big]E^{(\nu)}_{rs}(t)+\\&\quad+\varepsilon\int_0^\infty K_{klrs}\Big[\overset{\infty}{\underset{\xi=0}{G_{pq}(t-\xi)}};G^{(\mu)}_{pq},\tau\Big]E_{rs}(t-\tau)\,d\tau.\end{aligned} \tag{4.11}$$

The constitutive equation for small deformations superimposed on large deformations is now obtained by using Eqs. (4.3) and (4.11) in (4.2), again neglecting terms of order ε^2. Since ε was introduced only to facilitate the linearization process, it will be omitted after this step; *i.e.* we take $\varepsilon=1$ with the stipulation that the displacement gradients and their derivatives are small. The linearized equation is then

$$\begin{aligned}\sigma_{ij}(t)&=\Big[\frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_l}+\frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_l}+\frac{\partial x_i(t)}{\partial X_k}\frac{\partial u_j(t)}{\partial X_l}\Big]F_{kl}\Big[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}}\Big]+\\&\quad+\frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_l}\sum_{\nu}A^{(\nu)}_{klrs}\Big[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}};G^{(\mu)}_{pq}\Big]E^{(\nu)}_{rs}(t)+\\&\quad+\frac{\partial x_i(t)}{\partial X_k}\frac{\partial x_j(t)}{\partial X_l}\int_0^\infty K_{klrs}\Big[\overset{\infty}{\underset{\xi=0}{G_{pq}(t-\xi)}};G^{(\mu)}_{pq},\tau\Big]E_{rs}(t-\tau)\,d\tau.\end{aligned} \tag{4.12}$$

Since the stress tensor is symmetric, then $F_{kl}=F_{lk}$, $A^{(\nu)}_{klrs}=A^{(\nu)}_{lkrs}$, and $K_{klrs}=K_{lkrs}$. Since $E^{(\nu)}_{rs}=E^{(\nu)}_{sr}$, there is no loss of generality in taking $A^{(\nu)}_{klrs}=A^{(\nu)}_{klsr}$ and $K_{klrs}=K_{klsr}$ as well.

5. Materials with fading memory

If the basic deformation $x_i(X_p,t)$ were time-independent, the functionals F_{ij}, $A^{(\nu)}_{ijkl}$, and K_{ijkl} in Eq. (4.12) would reduce to ordinary functions of the constant strain components G_{pq}. The basic deformation is, of course, necessarily time-dependent, because it must involve a transition from the undeformed state to whatever fixed state is contemplated. However, the same simplification is achieved if a material with fading memory is held in a fixed state of deformation for so long that the original deforming process is forgotten.

Let the basic deformation $x_i(X_p,t)$ in Eq. (4.1) be one in which the body remains in a fixed state $x_i(X_p)$ after time $t=0$; *i.e.*

$$x_i(X_p,t)=x_i(X_p),\qquad t>0. \tag{5.1}$$

Using the notation $G_{pq}(t)$ for the strain histories, as before, let G_{pq} denote the constant strains associated with the fixed deformation $x_i(X_p)$. Since $G_{pq}(t)=G_{pq}$ for $t>0$, then the derivatives $G^{(\mu)}_{pq}$ $(\mu \neq 0)$ in Eq. (4.12) all vanish.

We assume that after a sufficient amount of time has elapsed, the deformation history before $t=0$ has an arbitrarily small effect on the stress. This assumption requires that the following condition be satisfied by the functionals F_{kl} which appear in Eq. (4.2):

$$\lim_{t\to\infty}\Big\{F_{ij}\Big[\mathop{G_{pq}(t-\tau)}_{\tau=0}^{\infty}+2E_{pq}(t-\tau)\Big]-F_{ij}\Big[G_{pq}+2\mathop{E_{pq}(t-\tau)}_{\tau=0}^{\infty}\Big]\Big\}=0. \qquad (5.2)$$

As a particular case of Eq. (5.2), we have

$$\lim_{t\to\infty} F_{ij}\Big[\mathop{G_{pq}(t-\tau)}_{\tau=0}^{\infty}\Big]=F_{ij}\Big[\mathop{G_{pq}}_{\tau=0}^{\infty}\Big]=F_{ij}(G_{pq}) \quad \text{(say)}. \qquad (5.3)$$

As indicated, the functionals reduce to functions of the constant values G_{pq} in this case. By considering the linearized form of Eq. (5.2), *i.e.* the form corresponding to Eq. (4.12), it can be shown that the functionals $A^{(\nu)}_{ijkl}$ and K_{ijkl} also reduce to functions of G_{pq} at large times.

When the deformation has been held fixed long enough that the history before $t=0$ has no perceptible effect, the functionals F_{ij}, $A^{(\nu)}_{ijkl}$, and K_{ijkl} may be replaced by their limiting forms, which are functions. The constitutive equation (4.12) then assumes the form

$$\begin{aligned}\sigma_{ij}(t)=&\left[\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_l}+\frac{\partial u_i(t)}{\partial X_k}\frac{\partial x_j}{\partial X_l}+\frac{\partial x_i}{\partial X_k}\frac{\partial u_j(t)}{\partial X_l}\right]F_{kl}(G_{pq})+\\&+\frac{\partial x_i}{\partial X_k}\frac{\partial x_j}{\partial X_l}\left[\sum_{\nu}A^{(\nu)}_{klrs}(G_{pq})\,E^{(\nu)}_{rs}(t)+\int_0^\infty K_{klrs}(G_{pq},\tau)\,E_{rs}(t-\tau)\,d\tau\right].\end{aligned} \qquad (5.4)$$

Here x_i denotes the basic steady deformation, with associated strain components G_{pq}.

In passing, we note that if u_i is steady, then the combination x_i+u_i can be regarded as a new steady deformation with no unsteady part superimposed. The stress given by Eq. (5.4) must be the same, whichever way the deformation is regarded. From this consideration, we find after some manipulation that the following relation must be satisfied:

$$A^{(0)}_{ijkl}(G_{pq})+\int_0^\infty K_{ijkl}(G_{pq},\tau)\,d\tau=2\frac{\partial F_{ij}(G_{pq})}{\partial G_{kl}}. \qquad (5.5)$$

In the remainder of this paper we will consider only those materials and deformations for which Eq. (5.4) is valid.

6. An alternative approach to materials with fading memory

In Sections 2 to 5 we have developed the constitutive equation for small dynamic deformations superimposed on large steady deformations in materials with fading memory. This was done by specializing the constitutive equation for materials with an unspecified type of memory. An alternative approach is possible, in which it is assumed initially that the deformation consists of a steady deformation $x_i(X_p)$ with a displacement field $u_i(X_p, t)$ superimposed on it. It is then assumed that the stress components depend on the deformation gradients $\partial x_p/\partial X_q$ and the history of the displacement gradients $\partial u_p(t)/\partial X_q$, so that

$$\sigma_{ij}(t) = f_{ij}[\partial u_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q;\ \partial x_p/\partial X_q]. \tag{6.1}$$

To find the restricted forms of f_{ij} which will satisfy the rotation invariance principle, we consider a new total deformation $\bar{x}_i(t) = \bar{x}_i + \bar{u}_i(t)$ related to the original total deformation $x_i(t) = x_i + u_i(t)$ by a rigid rotation, *i.e.* by Eq. (3.1). The stress components corresponding to the new deformation are given by

$$\bar{\sigma}_{ij}(t) = f_{ij}[\partial \bar{u}_p(t - \tau)/\partial X_q;\ \partial \bar{x}_p/\partial X_q]. \tag{6.2}$$

The new stress components $\bar{\sigma}_{ij}(t)$ must then be related to the components $\sigma_{ij}(t)$ by the transformation (3.5). By using Eqs. (6.1) and (6.2) in Eq. (3.5), and making use of Eq. (3.2), we obtain

$$f_{ij}[\partial u_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q;\ \partial x_p/\partial X_q] = a_{ki}(t)\, a_{lj}(t)\, f_{kl}[\partial \bar{u}_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q;\ \partial \bar{x}_p/\partial X_q]. \tag{6.3}$$

At this point, neither $\bar{x}_i$ nor $\bar{u}_i(t)$ has been defined. Only their sum, $\bar{x}_i(t)$, has been defined by (3.1). The new total deformation $\bar{x}_i(t)$ can be decomposed into a steady part and an unsteady part in either of two essentially different ways: (i) the steady part $\bar{x}_i$ may be regarded as identical with the steady part x_i of the original deformation, in which case the entire rigid motion is included in the unsteady part $\bar{u}_i(t)$, (ii) the new steady part $\bar{x}_i$ may be related to x_i by a time-independent rigid rotation, with $\bar{u}_i(t)$ related to $u_i(t)$ by the same rotation.

In case (i), we take $\bar{x}_i = x_i$. The displacement field $\bar{u}_i(t)$ is then defined by

$$\bar{u}_i(t) = \bar{x}_i(t) - \bar{x}_i = a_{ij}(t)\, x_j(t) - x_i. \tag{6.4}$$

Equation (6.3) then becomes

$$\begin{aligned} &f_{ij}[\partial u_p(t \underset{\tau=0}{\overset{\infty}{-}} \tau)/\partial X_q;\ \partial x_p/\partial X_q] \\ &\quad = a_{ki}(t)\, a_{lj}(t)\, f_{kl}[a_{pr}(t-\tau)\, \underset{\tau=0}{\overset{\infty}{\partial x_r}}(t-\tau)/\partial X_q - \partial x_p/\partial X_q;\ \partial x_p/\partial X_q]. \end{aligned} \tag{6.5}$$

By constructing the coefficients $a_{ij}(t)$ from the gradients $\partial x_i(t)/\partial X_j$, as in Section 3, we find that f_{ij} must be expressible in the form

$$f_{ij} = \frac{\partial x_i(t)}{\partial X_k}\,\frac{\partial x_j(t)}{\partial X_l}\, F^*_{kl}[G_{pq}(t \underset{\tau=0}{\overset{\infty}{-}} \tau);\ \partial x_p/\partial X_q]. \tag{6.6}$$

When Eq. (6.6) is used in the basic invariance requirement (6.3), and Eqs. (3.1) and (3.2) are used to simplify the resulting expression, it is found that the functionals F^*_{ij} must satisfy the condition

$$F^*_{ij}[G_{pq}(t \underset{\tau=0}{\overset{\infty}{-}} \tau);\ \partial x_p/\partial X_q] = F^*_{ij}[\bar{G}_{pq}(t \underset{\tau=0}{\overset{\infty}{-}} \tau);\ \partial \bar{x}_p/\partial X_q], \tag{6.7}$$

where the strain components $\bar{G}_{pq}(t)$ are defined by replacing $x_i(t)$ by $\bar{x}_i(t)$ in Eq. (3.8). By making use of Eqs. (3.1) and (3.2), it is found that

$$\bar{G}_{pq}(t) = G_{pq}(t). \tag{6.8}$$

We now consider case (ii). The coefficients a_{ij} in Eqs. (3.1) and (3.2) are taken to be constants, and the new steady deformation $\bar{x}_i$ is defined by

$$\bar{x}_i = a_{ij}\, x_j. \tag{6.9}$$

By using Eqs. (6.8) and (6.9) in Eq. (6.7), we find that the functionals F_{ij}^* must be of such form that

$$F_{ij}^*\,[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}};\, \partial x_p/\partial X_q] = F_{ij}^*\,[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}};\, a_{pr}\, \partial x_r/\partial X_q]. \tag{6.10}$$

By constructing the coefficients a_{ij} from the steady deformation gradients $\partial x_p/\partial X_q$ in the manner described in Section 3, we find that the functionals F_{ij}^* depend on the gradients $\partial x_p/\partial X_q$ only through the strain components G_{pq} defined in terms of them. By using this result in Eq. (6.6), and the result so obtained in Eq. (6.1), we obtain finally

$$\sigma_{ij}(t) = \frac{\partial x_i(t)}{\partial X_k}\,\frac{\partial x_j(t)}{\partial X_l}\, F_{kl}[\overset{\infty}{\underset{\tau=0}{G_{pq}(t-\tau)}};\, G_{pq}]. \tag{6.11}$$

The displacement gradients $\partial u_i(t)/\partial X_j$ enter into Eq. (6.11) through $\partial x_i(t)/\partial X_j$ and $G_{pq}(t-\tau)$. In general, the displacements $u_i(t-\tau)$ cannot be small throughout the entire history of deformation, since $x_i + u_i(t-\tau) = X_i$ for τ sufficiently large. Thus, before Eq. (6.11) can be linearized with respect to the displacement gradients it is necessary to assume that the material has a fading memory and that enough time has elapsed so that the deforming process has been forgotten. With these assumptions, Eq. (6.11) can be linearized, and the form obtained is again given by Eq. (5.4).

7. Symmetry of the material

Certain restrictions must be placed on the functions F_{ij}, $A_{ijkl}^{(\nu)}$, and K_{ijkl} in (5.4), if the constitutive equation is to describe a material which has some symmetry in its undeformed state. Let $\bar{x}$ be a fixed rectangular Cartesian coordinate system, related to the system x by an orthogonal transformation, so that

$$\bar{x}_i = s_{ij}\, x_j, \qquad \bar{X}_i = s_{ij}\, X_j, \qquad \bar{u}_i(t) = s_{ij}\, u_j(t), \tag{7.1}$$

where

$$s_{ik}\, s_{jk} = s_{ki}\, s_{kj} = \delta_{ij}. \tag{7.2}$$

Defining $\bar{G}_{pq}$, $\bar{E}_{pq}(t)$, and $\bar{E}_{pq}^{(\nu)}(t)$ as in Eqs. (3.8), (4.5), and (4.9) respectively, but in terms of the new coordinates $\bar{x}_i$ and $\bar{X}_i$, and making use of Eqs. (7.1) and (7.2), we obtain

$$\bar{G}_{ij} = s_{ik}\, s_{jl}\, G_{kl}, \qquad \bar{E}_{ij}^{(\nu)}(t) = s_{ik}\, s_{jl}\, E_{kl}^{(\nu)}(t). \tag{7.3}$$

The components $\bar{\sigma}_{ij}(t)$ of stress, measured with respect to the system $\bar{x}$, are given by

$$\bar{\sigma}_{ij}(t) = s_{ik}\, s_{jl}\, \sigma_{kl}(t). \tag{7.4}$$

If $\mathbf{s} = \|s_{ij}\|$ is a symmetry transformation for the material, then Eq. (5.4) remains valid if all quantities are measured with respect to the system $\bar{x}$ instead of the system x. By using Eq. (5.4) and the corresponding expression for $\bar{\sigma}_{ij}(t)$

to eliminate $\sigma_{ij}(t)$ and $\bar{\sigma}_{ij}(t)$ from Eq. (7.4), a relation involving F_{ij}, $A^{(\nu)}_{ijkl}$, and K_{ijkl} is obtained. From that relation and Eqs. (7.1) to (7.3), it can be shown after some manipulation that the following conditions must be satisfied (for details of a similar manipulation in the isotropic case, see [*1*] or [*2*]):

$$F_{ij}(\bar{G}_{pq}) = s_{ik}\, s_{jl}\, F_{kl}(G_{pq}), \tag{7.5}$$

$$A^{(\nu)}_{ijrs}(\bar{G}_{pq})\, \bar{E}^{(\nu)}_{rs}(t) = s_{ik}\, s_{jl}\, A^{(\nu)}_{klrs}(G_{pq})\, E^{(\nu)}_{rs}(t), \tag{7.6}$$

$$K_{ijrs}(\bar{G}_{pq}, \tau)\, \bar{E}_{rs}(t-\tau) = s_{ik}\, s_{jl}\, K_{klrs}(G_{pq}, \tau)\, E_{rs}(t-\tau). \tag{7.7}$$

These relations must be satisfied identically for each transformation $\mathbf{s}$ belonging to the group of symmetries of the material. In the following section we will consider the consequences of Eqs. (7.5) to (7.7) in the case of isotropic materials.

8. Isotropic materials

The symmetry group for an isotropic material is either the full orthogonal group or the proper orthogonal group, depending on whether the material has a center of symmetry or not. Presence or absence of a center of symmetry is irrelevant in the present context, however, because Eqs. (7.5) to (7.7) are satisfied identically if the transformation is the central inversion, *i.e.* if $s_{ij} = -\delta_{ij}$.

RIVLIN & ERICKSEN [*4*] have studied the implications of an equation of the form (7.5). They have shown that if the functions F_{ij} are single-valued functions of the variables G_{pq}, and if Eq. (7.5) is satisfied for every orthogonal transformation $\mathbf{s}$, then the matrix $\boldsymbol{F} = \|F_{ij}\|$ can be expressed in terms of the matrix $\boldsymbol{G} = \|G_{ij}\|$ in the form

$$\boldsymbol{F} = F_0\,\boldsymbol{I} + F_1\,\boldsymbol{G} + F_2\,\boldsymbol{G}^2. \tag{8.1}$$

Here $\boldsymbol{I}$ is the unit matrix $\|\delta_{ij}\|$. The scalar coefficients F_0, F_1 and F_2 are single-valued functions of three basic orthogonal invariants of $\boldsymbol{G}$,

$$\operatorname{tr}\boldsymbol{G} = G_{ii}, \qquad \operatorname{tr}\boldsymbol{G}^2 = G_{ij}\,G_{ij}, \qquad \operatorname{tr}\boldsymbol{G}^3 = G_{ij}\,G_{jk}\,G_{ki}. \tag{8.2}$$

If, in (7.5), F_{ij} is a polynomial* in its arguments, then $\boldsymbol{F}$ may be expressed in the form (8.1), with F_0, F_1, and F_2 expressible as polynomials in the invariants (8.2).

The limitations on $K_{ijrs}\,E_{rs}(t-\tau)$ implied by (7.7) may be obtained in more explicit form by considerations of the type used by RIVLIN & ERICKSEN [*4*] and by RIVLIN [*5*]. We use the notation

$$K_{ij}(G_{pq}, E_{rs}) = K_{ijrs}(G_{pq})\, E_{rs}, \tag{8.3}$$

the dependence of E_{rs} on $t-\tau$ being understood. Then, Eq. (7.7) may be written in the form

$$K_{ij}(\bar{G}_{pq}, \bar{E}_{rs}) = s_{ik}\, s_{jl}\, K_{kl}(G_{pq}, E_{rs}). \tag{8.4}$$

If we assume that K_{ijrs} is a polynomial in G_{pq}, so that K_{ij} is a polynomial in G_{pq} and E_{rs}, linear in E_{rs}, it follows (RIVLIN [*5*]) that $\boldsymbol{K} = \|K_{ij}\|$ may be expressed in the form

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* We note that if F_{ij} is a continuous function of G_{pq}, it may be expressed as a polynomial as closely as desired in any closed domain.

$$\boldsymbol{K} = K_0 \boldsymbol{E} + K_1 (\boldsymbol{E}\,\boldsymbol{G} + \boldsymbol{G}\,\boldsymbol{E}) + K_2 (\boldsymbol{E}\,\boldsymbol{G}^2 + \boldsymbol{G}^2 \boldsymbol{E}) + \sum_{M,N=0}^{2} K_{MN} (\operatorname{tr} \boldsymbol{E}\,\boldsymbol{G}^N)\, \boldsymbol{G}^M. \tag{8.5}$$

The coefficients K_0, K_1, K_2 and K_{MN} are polynomials in the invariants (8.2), and functions of τ. If these coefficients are regarded as single-valued functions rather than polynomials, Eq. (8.4) is still satisfied.

Since Eq. (7.6) is of the same form as Eq. (7.7), its implications are analogous to those of (7.7). We use the notation

$$A_{ij}^{(\nu)} (G_{pq}, E_{rs}^{(\nu)}) = A_{ijrs}^{(\nu)} (G_{pq})\, E_{rs}^{(\nu)} (t). \tag{8.6}$$

By an argument similar to that used in deriving (8.5), we find that $\boldsymbol{A}^{(\nu)} = \| A_{ij}^{(\nu)} \|$ can be expressed in the form

$$\boldsymbol{A}^{(\nu)} = A_0^{(\nu)} \boldsymbol{E}^{(\nu)} + A_1^{(\nu)} (\boldsymbol{E}^{(\nu)} \boldsymbol{G} + \boldsymbol{G}\,\boldsymbol{E}^{(\nu)}) + A_2^{(\nu)} (\boldsymbol{E}^{(\nu)} \boldsymbol{G}^2 + \boldsymbol{G}^2 \boldsymbol{E}^{(\nu)}) + \sum_{M,N=0}^{2} A_{NM}^{(\nu)} (\operatorname{tr} \boldsymbol{E}^{(\nu)} \boldsymbol{G}^N)\, \boldsymbol{G}^M. \tag{8.7}$$

The coefficients $A_0^{(\nu)}$, $A_1^{(\nu)}$, $A_2^{(\nu)}$, and $A_{MN}^{(\nu)}$ are polynomials in the invariants (8.2). Again, the invariance requirements are still satisfied if these coefficients are regarded as single-valued functions rather than polynomials.

When Eq. (8.5) is used in (8.3), (8.7) is used in (8.6), and (8.1), (8.3), and (8.6) are then used in Eq. (5.4), the constitutive equation for isotropic materials is obtained. It may be written as

$$\begin{aligned}
\sigma_{ij}(t) = & \left[\frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_l} + \frac{\partial u_i(t)}{\partial X_k} \frac{\partial x_j}{\partial X_l} + \frac{\partial x_i}{\partial X_k} \frac{\partial u_j(t)}{\partial X_l} \right] [F_0\, \delta_{kl} + F_1\, G_{kl} + F_2\, G_{kp}\, G_{pl}] + \\
& + \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_l} \sum_{\nu} \Big[A_0^{(\nu)}\, \delta_{kp}\, \delta_{lq} + A_1^{(\nu)} (\delta_{kp}\, G_{lq} + \delta_{lq}\, G_{kp}) + \\
& + A_2^{(\nu)} (\delta_{kp}\, G_{lr}\, G_{rq} + \delta_{lq}\, G_{kr}\, G_{rp}) + \sum_{M,N=0}^{2} A_{MN}^{(\nu)} (\boldsymbol{G}^M)_{kl} (\boldsymbol{G}^N)_{pq} \Big] E_{pq}^{(\nu)}(t) + \\
& + \frac{\partial x_i}{\partial X_k} \frac{\partial x_j}{\partial X_l} \int_0^\infty \Big[K_0(\tau)\, \delta_{kp}\, \delta_{lq} + K_1(\tau)\, (\delta_{kp}\, G_{lq} + \delta_{lq}\, G_{kp}) + \\
& + K_2(\tau)\, (\delta_{kp}\, G_{lr}\, G_{rq} + \delta_{lq}\, G_{kr}\, G_{rp}) + \sum_{M,N=0}^{2} K_{MN}(\tau)\, (\boldsymbol{G}^M)_{kl} (\boldsymbol{G}^N)_{pq} \Big] E_{pq}(t-\tau)\, d\tau.
\end{aligned} \tag{8.8}$$

Dependence of the scalar coefficients on tr $\boldsymbol{G}^N$ ($N = 1, 2, 3$) is understood.

9. An alternative form for the constitutive equation for isotropic materials

Equation (8.8) can be written in a slightly simpler form. As the first step, the displacement gradients are written in terms of spatial derivatives, thus

$$\frac{\partial u_i(t)}{\partial X_j} = \frac{\partial u_i(t)}{\partial x_k} \frac{\partial x_k}{\partial X_j}. \tag{9.1}$$

Here the derivatives $\partial x_k / \partial X_j$ are deformation gradients for the basic steady deformation. By using (9.1) in the definition (4.5) of E_{pq}, we obtain

$$E_{pq}(t) = \frac{\partial x_i}{\partial X_p} \frac{\partial x_j}{\partial X_q} e_{ij}(t), \tag{9.2}$$

where e_{ij} is the classical strain tensor defined by

$$e_{ij}(t) = \frac{1}{2}\left[\frac{\partial u_i(t)}{\partial x_j} + \frac{\partial u_j(t)}{\partial x_i}\right]. \tag{9.3}$$

The time-derivatives $E^{(\mu)}_{pq}$ are, from (9.2),

$$E^{(\mu)}_{pq}(t) = \frac{\partial x_i}{\partial X_p}\,\frac{\partial x_j}{\partial X_q}\,e^{(\mu)}_{ij}(t), \tag{9.4}$$

where

$$e^{(\mu)}_{ij}(t) = d^{\mu}\,e_{ij}(t)/d t^{\mu}. \tag{9.5}$$

In the definition (4.9) of $E^{(\mu)}_{pq}$, and now in Eq. (9.5), the differentiation d/dt holds X_p fixed. Thus

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial u_i(t)}{\partial t}\,\frac{\partial}{\partial x_i}. \tag{9.6}$$

When this expression is used in (9.5), terms arise which are quadratic in the displacements. These will be neglected. Thus, in (9.5), d/dt may be regarded as the partial derivative with x_i held fixed.

Next, consider the tensor $\boldsymbol{g} = \|g_{ij}\|$, defined by

$$g_{ij} = \frac{\partial x_i}{\partial X_p}\,\frac{\partial x_j}{\partial X_p}. \tag{9.7}$$

By using this definition with Eqs. (9.2), (9.4), and the definition (3.8) of G_{pq}, we obtain the relations

$$\operatorname{tr}\boldsymbol{G}^N = \operatorname{tr}\boldsymbol{g}^N, \qquad \operatorname{tr}\,(\boldsymbol{E}^{(\nu)}\,\boldsymbol{G}^N) = \operatorname{tr}(\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}^{N+1}), \tag{9.8}$$

$$\frac{\partial x_i}{\partial X_k}\,\frac{\partial x_j}{\partial X_l}\,(\boldsymbol{G}^N)_{kl} = (\boldsymbol{g}^{N+1})_{ij}, \tag{9.9}$$

$$\frac{\partial x_i}{\partial X_k}\,\frac{\partial x_j}{\partial X_l}\,(\boldsymbol{E}^{(\nu)}\,\boldsymbol{G}^N)_{kl} = (\boldsymbol{g}\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}^{N+1})_{ij}, \tag{9.10}$$

$$\begin{aligned}\left[\frac{\partial x_i}{\partial X_k}\,\frac{\partial x_j}{\partial X_l} + \frac{\partial u_i(t)}{\partial X_k}\,\frac{\partial x_j}{\partial X_l} + \frac{\partial x_i}{\partial X_k}\,\frac{\partial u_j(t)}{\partial X_l}\right](\boldsymbol{G}^N)_{kl}&\\ = [\delta_{ik}\,\delta_{jl} + u_{i,k}(t)\,\delta_{jl} + u_{j,l}(t)\,\delta_{ik}]\,(\boldsymbol{g}^{N+1})_{kl}.&\end{aligned} \tag{9.11}$$

In the last equation, the notation $\partial u_i/\partial x_j = u_{i,j}$ has been used.

By making use of Eqs. (9.8) to (9.11), the constitutive equation (8.8) can be written in the form

$$\begin{aligned}\sigma_{ij}(t) = {}& [\delta_{ik}\,\delta_{jl} + u_{i,k}(t)\,\delta_{jl} + u_{j,l}(t)\,\delta_{ik}]\,(F_0\,\boldsymbol{g} + F_1\,\boldsymbol{g}^2 + F_2\,\boldsymbol{g}^3)_{kl} + \\ & + \sum_{\nu}\Big[A^{(\nu)}_0\,\boldsymbol{g}\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g} + A^{(\nu)}_1\,(\boldsymbol{g}\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}^2 + \boldsymbol{g}^2\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}) + \\ & + A^{(\nu)}_2\,(\boldsymbol{g}\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}^3 + \boldsymbol{g}^3\,\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}) + \sum_{M,N=0}^{2} A^{(\nu)}_{MN}\operatorname{tr}(\boldsymbol{e}^{(\nu)}\,\boldsymbol{g}^{N+1})\,\boldsymbol{g}^{M+1}\Big]_{ij} + \\ & + \int_0^{\infty}\Big[K_0(\tau)\,\boldsymbol{g}\,\boldsymbol{e}\,\boldsymbol{g} + K_1(\tau)\,(\boldsymbol{g}\,\boldsymbol{e}\,\boldsymbol{g}^2 + \boldsymbol{g}^2\,\boldsymbol{e}\,\boldsymbol{g}) + \\ & + K_2(\tau)\,(\boldsymbol{g}\,\boldsymbol{e}\,\boldsymbol{g}^3 + \boldsymbol{g}^3\,\boldsymbol{e}\,\boldsymbol{g}) + \sum_{M,N=0}^{2} K_{MN}(\tau)\operatorname{tr}(\boldsymbol{e}\,\boldsymbol{g}^{N+1})\,\boldsymbol{g}^{M+1}\Big]_{ij}\,d\tau.\end{aligned} \tag{9.12}$$

The time-dependence of $\boldsymbol{e}$ and $\boldsymbol{e}^{(\nu)}$ is understood. According to (9.8), the scalar coefficients in (9.12) may be regarded as functions of $\operatorname{tr}\boldsymbol{g}^N$ $(N=1, 2, 3)$ rather than of $\operatorname{tr}\boldsymbol{G}^N$ $(N=1, 2, 3)$.

The integration with respect to time in (9.12) is integration with X_p held fixed. However, with neglect of terms quadratic in the displacements, it may be regarded as an integration holding x_i fixed.

By using the Cayley-Hamilton theorem, $\boldsymbol{g}^3$ may be expressed as a linear combination of $\boldsymbol{g}^2, \boldsymbol{g}$, and $\boldsymbol{I}$, with coefficients which are polynomials in the invariants $\operatorname{tr} \boldsymbol{g}^N$ $(N=1, 2, 3)$. Therefore we may write

$$F_0 \boldsymbol{g} + F_1 \boldsymbol{g}^2 + F_2 \boldsymbol{g}^3 = f_0 \boldsymbol{I} + f_1 \boldsymbol{g} + f_2 \boldsymbol{g}^2, \tag{9.13}$$

where f_0, f_1, and f_2 are functions of $\operatorname{tr} \boldsymbol{g}^N$ $(N=1, 2, 3)$.

Denoting the integrand in Eq. (9.12) by k_{ij} and its matrix by $\boldsymbol{k} = \|k_{ij}\|$, we see that $\boldsymbol{k}$ is a symmetric matrix polynomial in the symmetric matrices $\boldsymbol{g}$ and $\boldsymbol{e}$, linear in $\boldsymbol{e}$. The scalar coefficients are, of course, not necessarily polynomials. According to Rivlin [*5*], $\boldsymbol{k}$ may therefore be expressed in a form similar to (8.5). In that expression we replace $\boldsymbol{K}$ by $\boldsymbol{k}$, $\boldsymbol{G}$ by $\boldsymbol{g}$ and $\boldsymbol{E}$ by $\boldsymbol{e}$ to get the desired result. Similarly, the summand in (9.12) can be expressed in a form analogous to Eq. (8.7), by replacing $\boldsymbol{G}$ by $\boldsymbol{g}$ and $\boldsymbol{E}$ by $\boldsymbol{e}$ in the latter expression. These results, taken with Eq. (9.13), imply that Eq. (9.12) can be reduced to the following form:

$$\begin{aligned} \sigma_{ij}(t) = {} & [\delta_{ik}\,\delta_{jl} + u_{i,k}(t)\,\delta_{jl} + u_{j,l}(t)\,\delta_{ik}]\,[f_0 \boldsymbol{I} + f_1 \boldsymbol{g} + f_2 \boldsymbol{g}^2]_{kl} + \\ & + \sum_{\nu} \Big[a_0^{(\nu)}\,\delta_{ik}\,\delta_{jl} + a_1^{(\nu)}(\delta_{ik}\,g_{jl} + \delta_{jl}\,g_{ik}) + \\ & + a_2^{(\nu)}(\delta_{ik}\,g_{jn}\,g_{nl} + \delta_{jl}\,g_{in}\,g_{nk}) + \\ & + \sum_{M,N=0}^{2} a_{MN}^{(\nu)}(\boldsymbol{g}^M)_{ij}(\boldsymbol{g}^N)_{kl}\Big]\, e_{kl}^{(\nu)}(t) + \\ & + \int_0^\infty \Big[k_0(\tau)\,\delta_{ik}\,\delta_{jl} + k_1(\tau)\,(\delta_{ik}\,g_{jl} + \delta_{jl}\,g_{ik}) + \\ & + k_2(\tau)\,(\delta_{ik}\,g_{jn}\,g_{nl} + \delta_{jl}\,g_{in}\,g_{nk}) + \\ & + \sum_{M,N=0}^{2} k_{MN}(\tau)\,(\boldsymbol{g}^M)_{ij}(\boldsymbol{g}^N)_{kl}\Big]\, e_{kl}(t-\tau)\,d\tau. \end{aligned} \tag{9.14}$$

The scalar coefficients are functions of $\operatorname{tr} \boldsymbol{g}^N$ $(N=1, 2, 3)$.

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A Generalised Rayleigh Quotient Iteration for Lambda-Matrices

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1. Introduction

The purpose of this paper is to present a new method for calculating the latent roots and vectors of a matrix* whose elements are polynomials in a scalar, λ. We refer to such a matrix as a λ-matrix, and consider the typical form:

$$D_l(\lambda) \equiv A_0 \lambda^l + A_1 \lambda^{l-1} + \cdots + A_{l-1} \lambda + A_l, \tag{1}$$

where the A_r ($r = 1, 2, \ldots, l$) are matrices of the same order with real or complex elements. The latent roots, λ_i, and the right and left latent vectors $\boldsymbol{q}_i$, $\boldsymbol{r}_i$ are defined so that

$$D_l(\lambda_i)\, \boldsymbol{q}_i = 0 \quad \text{and} \quad D_l'(\lambda_i)\, \boldsymbol{r}_i = 0, \tag{2}$$

where D_l' denotes the transpose of D_l.

We define a *regular* λ-matrix as a square λ-matrix, of order n say, for which A_0 is non-singular. It is easily seen that such a λ-matrix has ln latent roots if they are counted according to their multiplicities.

We next define a *simple* λ-matrix as a regular λ-matrix such that for any latent root λ_i whose multiplicity is α, the constant matrix $D_l(\lambda_i)$ has degeneracy α (or rank $n-\alpha$), and also $\alpha \leqq n$ for all i.

This definition implies that there are α *linearly independent* vectors $\boldsymbol{q}_i$ associated with λ_i, and similarly for the $\boldsymbol{r}_i$. It can also be shown that it implies, and is implied by, the condition that λ_i has only *linear* elementary divisors. The former property has more significance in this application.

It is well known that the latent roots and vectors of $D_l(\lambda)$ can be found by considering those of an $ln \times ln$ matrix whose elements are linear in λ. See for example FRAZER, DUNCAN & COLLAR [*1*], and Section 2 below. For such matrices well established methods of solution are available. In particular, OSTROWSKI has given detailed proofs for the convergence of iterative methods based on use of the *Rayleigh quotient* in a series of papers listed under reference [*2*]. These will be referred to in the following as OSTROWSKI [2_1], ..., [2_6]. It is our aim to generalise the Rayleigh quotient and two of OSTROWSKI's iterative processes in such a way that they can be applied to the $n \times n$ matrix $D_l(\lambda)$ rather than a corresponding $ln \times ln$ matrix, and thus obtain a possible reduction in the computational labour involved.

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* My use of the terms latent root and latent vector may not be very orthodox. It is not consistent with the usual phraseology in which we define the latent roots of a constant matrix A as the zeros of $|A - \lambda I|$. In this work we speak only of the latent roots of matrices dependent on a scalar parameter. Thus we may refer to *the latent roots of* $A - \lambda I$, while *the latent roots of* A will have no meaning.

Our derivation of the generalised Rayleigh quotient will involve a transformation to the equivalent linear form used by the author in [*3*], and the proof of convergence for simple matrices depends on the spectral form for $[D_l(\lambda)]^{-1}$ obtained there.

The generalisation of the classical Rayleigh quotient and its properties are obtained in Sections 2, 3 and 4. In Sections 5 and 7 we define and discuss the iterative process based on this generalisation. In Sections 8 and 9 we establish some local convergence properties for the iteration, and in Section 11 we give two numerical examples. The closely related process known as "broken iteration" is discussed in Section 10.

2. The Rayleigh quotient for matrices linear in λ

When $l=1$ we have the simple λ-matrix

$$D_1(\lambda) \equiv A_0 \lambda + A_1, \tag{3}$$

and there are two sets of n linearly independent vectors $\boldsymbol{q}_i$ and $\boldsymbol{r}_i$ obtained from (2). In the classical problem of small vibrations of a conservative mechanical system about a position of equilibrium A_0 and A_1 are real symmetric matrices, so that we may assume $\boldsymbol{q}_i = \boldsymbol{r}_i$. This case was considered by CRANDALL [*4*] and TEMPLE [*5*], and they both employ the Rayleigh quotient in the form (in our notation):

$$R(\boldsymbol{q}) = -\frac{\boldsymbol{q}' A_1 \boldsymbol{q}}{\boldsymbol{q}' A_0 \boldsymbol{q}}. \tag{4}$$

The important properties of this quotient are that when considered as a function of the components of $\boldsymbol{q}$, R has extreme values when $\boldsymbol{q}$ is any latent vector of $D_1(\lambda)$, say $\boldsymbol{q}_i$, and the corresponding extreme value of R is the latent root λ_i. (*E.g.* see GOULD [*6*].)

OSTROWSKI obtains the form (4) in his paper [2_2], but in paper [2_3] he makes an important extension to non-symmetric matrices; although this is in the case $A_0 = I$ (the unit matrix) only. In this case the sets of vectors $\boldsymbol{q}_i$ and $\boldsymbol{r}_i$ are distinct and the first generalisation of Rayleigh's quotient takes the form

$$R(\boldsymbol{q}, \boldsymbol{r}) = -\frac{\boldsymbol{r}' A_1 \boldsymbol{q}}{\boldsymbol{r}' \boldsymbol{q}}. \tag{5}$$

OSTROWSKI shows that $R(\boldsymbol{q}, \boldsymbol{r})$ *has a stationary value at* $\boldsymbol{q} = \boldsymbol{q}_i$, $\boldsymbol{r} = \boldsymbol{r}_i$, *for* $i = 1, 2, \ldots, n$; *and that* $R(\boldsymbol{q}_i, \boldsymbol{r}_i) = \lambda_i$.

The obvious generalisation to matrices of the form (3) is

$$R(\boldsymbol{q}, \boldsymbol{r}) = -\frac{\boldsymbol{r}' A_1 \boldsymbol{q}}{\boldsymbol{r}' A_0 \boldsymbol{q}}. \tag{6}$$

In the next section we will consider this quotient for a linear λ-matrix equivalent to $D_l(\lambda)$, and this will suggest a form of Rayleigh's quotient defined in terms of $D_l(\lambda)$ itself. The properties of R as stated for (5) carry over to our more general form (proved in Sections 3 and 4); and in particular, these properties are established for (6) as a special case.

However, we note here that the value of R given by (6) when evaluated at $\boldsymbol{q}=\boldsymbol{q}_i$ and $\boldsymbol{r}=\boldsymbol{r}_i$ follows immediately from the well-known biorthogonality property of the latent vectors. When the latent vectors are suitably normalised this takes the form:

$$\boldsymbol{r}_i' A_0 \boldsymbol{q}_j = \delta_{ij}, \qquad \boldsymbol{r}_i' A_1 \boldsymbol{q}_j = -\delta_{ij}\lambda_i. \tag{7}$$

When $l>1$ no such simple orthogonality property exists, simply because there are ln latent vectors $\boldsymbol{q}_i$ or $\boldsymbol{r}_i$ in an n-dimensional vector space (*v.* Section 3 of [3]). Therefore any method for linear matrices employing this property cannot be immediately generalised to problems in which $l>1$.

3. The generalised Rayleigh quotient

For any latent vector $\boldsymbol{x}$ of either of equations (2) we denote by $\boldsymbol{x}^{(s)}$ the product $\lambda^s\boldsymbol{x}$, λ being the corresponding latent root. Then it can easily be verified that equation (2) is equivalent to the following $ln\times ln$ partitioned matrix equation whose elements are linear functions of λ:

$$\left\{\left[\begin{array}{cccccc} 0 & 0 & \cdots & & 0 & A_0 \\ 0 & 0 & \cdots & & A_0 & A_1 \\ & & & A_0 & A_1 & A_2 \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & A_0 & & & \\ 0 & A_0 & A_1 & \cdots & & . \\ A_0 & A_1 & A_2 & \cdots & . & A_{l-1} \end{array}\right]\lambda + \left[\begin{array}{ccccccc} 0 & 0 & \cdots & & 0 & -A_0 & 0 \\ 0 & 0 & \cdots & & -A_0 & -A_1 & 0 \\ \vdots & \vdots & & -A_0 & -A_1 & -A_2 & 0 \\ & & & & & \vdots & \vdots \\ 0 & -A_0 & & & & -A_{l-3} & 0 \\ -A_0 & -A_1 & \cdots & & -A_{l-3} & -A_{l-2} & 0 \\ 0 & 0 & \cdots & & 0 & 0 & A_l \end{array}\right]\right\}\times \tag{8}$$

$$\times\begin{bmatrix} \boldsymbol{q}^{(l-1)} \\ \boldsymbol{q}^{(l-2)} \\ \vdots \\ \boldsymbol{q}^{(1)} \\ \boldsymbol{q} \end{bmatrix} = 0.$$

Denote the two $ln\times ln$ matrices appearing in this equation by $\mathscr{A}$ and $\mathscr{C}$ respectively. It is clear that since A_0 is non-singular, so is $\mathscr{A}$; and that if all the matrices $A_0, A_1, \ldots, A_l$ are symmetric, then so are $\mathscr{A}$ and $\mathscr{C}$. Denote the $ln\times 1$ vector generated by λ and $\boldsymbol{q}$ by $\boldsymbol{q}^+$, and the corresponding vector generated by λ and $\boldsymbol{r}$ by $\boldsymbol{r}^+$.

Rayleigh's quotient in the form (6) now suggests that we investigate

$$R(\boldsymbol{q},\boldsymbol{r},\lambda) = -\frac{\boldsymbol{r}^{+\prime}\mathscr{C}\boldsymbol{q}^+}{\boldsymbol{r}^{+\prime}\mathscr{A}\boldsymbol{q}^+}. \tag{9}$$

For the numerator of (9) we have

$$\begin{aligned} \boldsymbol{r}^{+\prime}\mathscr{C}\boldsymbol{q}^+ &= [\lambda^{l-1}\boldsymbol{r}', \ldots, \lambda\boldsymbol{r}', \boldsymbol{r}']\begin{bmatrix} \\ \mathscr{C} \\ \\ \end{bmatrix}\begin{bmatrix} \lambda^{l-1}\boldsymbol{q} \\ \vdots \\ \lambda\boldsymbol{q} \\ \boldsymbol{q} \end{bmatrix} \\ &= \boldsymbol{r}'\{-(l-1)\lambda^l A_0 - (l-2)\lambda^{l-1}A_1 - \cdots - \lambda^2 A_{l-2} + A_l\}\boldsymbol{q} \qquad (10) \\ &= \boldsymbol{r}'\left\{-\lambda\frac{d}{d\lambda}D_l(\lambda) + D_l(\lambda)\right\}\boldsymbol{q}. \end{aligned}$$

And for the denominator of (9) we have

$$\begin{aligned}\boldsymbol{r}^{+\prime}\mathscr{A}\,\boldsymbol{q}^{+} &= \boldsymbol{r}'\{l\,\lambda^{l-1}A_0 + (l-1)\,\lambda^{l-2}A_1 + \cdots + 2\lambda\,A_{l-2} + A_{l-1}\}\,\boldsymbol{q}\\ &= \boldsymbol{r}'\left\{\frac{d}{d\lambda}D_l(\lambda)\right\}\boldsymbol{q}.\end{aligned} \tag{11}$$

Combining (9), (10) and (11) we obtain the following form for the Rayleigh quotient:

$$R(\boldsymbol{q},\boldsymbol{r},\lambda) = \lambda - \frac{\boldsymbol{r}'D_l(\lambda)\,\boldsymbol{q}}{\boldsymbol{r}'\left[\frac{d}{d\lambda}D_l(\lambda)\right]\boldsymbol{q}}, \tag{12}$$

and we consider this expression for all $\boldsymbol{q}$, $\boldsymbol{r}$, and λ, whether latent vectors and roots or not. $\left(\text{We assume always that } \boldsymbol{r}'\left[\frac{d}{d\lambda}D_l(\lambda)\right]\boldsymbol{q} \neq 0 \text{ for any set of latent values.}\right)$ The form (12) obviously has one of the usual properties of Rayleigh's quotient. That is, for latent vectors $\boldsymbol{q}_i$, $\boldsymbol{r}_i$ associated with the latent root λ_i,

$$R(\boldsymbol{q}_i,\boldsymbol{r}_i,\lambda_i) = \lambda_i. \tag{13}$$

4. The stationary property

We now prove that the function $R(\boldsymbol{q},\boldsymbol{r},\lambda)$ defined by (12) is such that R has a stationary value at $\boldsymbol{q}_i$, $\boldsymbol{r}_i$, λ_i, for $i=1, 2, \ldots, ln$. We may note in passing that the definition of R and the following proof do not depend on the polynomial nature of $D_l(\lambda)$. Thus, the same results will apply to any suitably defined matrix function of the scalar, λ — although they will only be of value if latent roots and vectors exist. However, we will proceed with no change in notation.

In order to establish the stationary property of R we make arbitrary small variations in $\boldsymbol{q}$, $\boldsymbol{r}$ and λ from a set of latent values, and show that the resulting change in R is zero to the first order. Thus, we write $\boldsymbol{q}=\boldsymbol{q}_i+\delta\boldsymbol{q}$, $\boldsymbol{r}=\boldsymbol{r}_i+\delta\boldsymbol{r}$, $\lambda=\lambda_i+\delta\lambda$, so that

$$R(\boldsymbol{q}_i,\boldsymbol{r}_i,\lambda_i) + \delta R = \lambda_i + \delta\lambda - \frac{(\boldsymbol{r}_i'+\delta\boldsymbol{r}')\,D_l(\lambda_i+\delta\lambda)\,(\boldsymbol{q}_i+\delta\boldsymbol{q})}{(\boldsymbol{r}_i'+\delta\boldsymbol{r}')\left[\frac{d}{d\lambda}D_l(\lambda)\right]_{\lambda_i+\delta\lambda}(\boldsymbol{q}_i+\delta\boldsymbol{q})}. \tag{14}$$

For the numerator of the quotient on the right we have

$$\begin{aligned}(\boldsymbol{r}_i'+\delta\boldsymbol{r}')\,D_l(\lambda_i+\delta\lambda)\,(\boldsymbol{q}_i+\delta\boldsymbol{q}) &\sim (\boldsymbol{r}_i'+\delta\boldsymbol{r}')\left\{D_l(\lambda_i)+\delta\lambda\left[\frac{d}{d\lambda}D_l(\lambda)\right]_{\lambda_i}\right\}(\boldsymbol{q}_i+\delta\boldsymbol{q}),\\ &\simeq \delta\lambda\,\boldsymbol{r}_i'\left[\frac{d}{d\lambda}D_l(\lambda)\right]_{\lambda_i}\boldsymbol{q}_i,\end{aligned} \tag{15}$$

to first order in the variations, using equations (2). For convenience we will in future denote $\left[\frac{d}{d\lambda}D_l(\lambda)\right]_{\lambda_i}$ by $D_l^{(1)}(\lambda_i)$. Having already assumed that the right-hand side of (15) does not vanish, we may combine this with (14) and (13); when it is found that $\delta R=0$ to first order in the variations; this being true for an arbitrary variation of $\boldsymbol{q}$, $\boldsymbol{r}$ and λ about any one of the ln sets of latent values. Hence the following statement:

The generalisation of Rayleigh's quotient, $R(\boldsymbol{q},\boldsymbol{r},\lambda)$, *given by* (12) *is such that* $R(\boldsymbol{q}_i,\boldsymbol{r}_i,\lambda_i)=\lambda_i$ *for* $i=1, 2, \ldots, ln$, *and* R *has a stationary value with respect to* λ *and the components of* $\boldsymbol{q}$ *and* $\boldsymbol{r}$ *at each of these points, provided that* $\boldsymbol{r}_i'D_l^{(1)}(\lambda_i)\,\boldsymbol{q}_i \neq 0$.

5. An iterative process

We suppose that we are given an approximation λ_0 to a latent root, λ, of D_l. We construct a sequence λ_s; $s=0, 1, 2, \ldots$, and two sequences of vectors, $\boldsymbol{\xi}_s$ and $\boldsymbol{\eta}_s$ by the following rules:

Using arbitrary vectors $\boldsymbol{w}$ *and* $\boldsymbol{z}$, *we form*

$$\boldsymbol{\xi}_s=[D_l(\lambda_s)]^{-1}\boldsymbol{w}, \quad \boldsymbol{\eta}_s=[D_l'(\lambda_s)]^{-1}\boldsymbol{z}; \tag{16}$$

and then

$$\begin{aligned}\lambda_{s+1}&=R(\boldsymbol{\xi}_s, \boldsymbol{\eta}_s, \lambda_s)\\ &=\lambda_s-\frac{\boldsymbol{\eta}_s' D_l(\lambda_s)\,\boldsymbol{\xi}_s}{\boldsymbol{\eta}_s' D_l^{(1)}(\lambda_s)\,\boldsymbol{\xi}_s}.\star\end{aligned} \tag{17}$$

Iterative processes of this type are discussed by OSTROWSKI [2_1] and by CRANDALL [4] for some special linear λ-matrices, and they both demonstrate that, provided $|\lambda_0-\lambda|$ is sufficiently small, the sequence λ_s tends towards the latent root λ; the convergence being *quadratic*. That is, if $|\lambda_s-\lambda|=\Delta$, then $|\lambda_{s+1}-\lambda|=O(\Delta^2)$. We will establish convergence of this order for the iterative process defined by (16) and (17) when applied to *simple* λ-matrices of any degree.

Although the vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ appearing in (16) are described as arbitrary, there is one condition to be satisfied by each of them, the reason for which will become clear in Section 7. If the sequence λ_s converges to the latent root λ, then there must exist at least one pair of latent vectors $\boldsymbol{q}$ and $\boldsymbol{r}$ associated with λ such that

$$\boldsymbol{r}'\boldsymbol{w}\neq 0 \quad \text{and} \quad \boldsymbol{q}'\boldsymbol{z}\neq 0. \tag{18}$$

In practice there will be zero probability of either of these conditions being violated, so the use of the word arbitrary may be excused.

An obvious difficulty arises if the matrices A_r of (1) are all real and we seek a latent root which is complex. If $\boldsymbol{w}$, $\boldsymbol{z}$, and λ_0 are all chosen to be real then the process breaks down, because the operations involved in (16) and (17) will all be real, so that the iterates can never converge to a complex set of latent values. In cases of doubt a complex λ_0 should therefore be chosen.

This iterative process may well be suitable for computation on a high-speed digital computer. The main computational difficulty may lie in the inversion of the ill-conditioned matrices, $D_l(\lambda_s)$.

6. Spectral form for the inverse of a simple λ-matrix

For future reference we will now state a theorem proved by the author in [3] which gives an expression for $[D_l(\lambda)]^{-1}$ in terms of all the latent roots and vectors of equations (2), provided D_l is simple, and provided λ itself is not a latent root. In the following, Q and R denote the $n\times ln$ matrices whose columns are the two sets of ln latent vectors of equations (2) written in some fixed order. The corresponding $ln\times ln$ *diagonal* matrix of the latent roots we call Λ.

If $\boldsymbol{q}_i$, $\boldsymbol{r}_i$, λ_i denote a set of latent values, we can rewrite a part of Theorem 2 of [3] in the following way, better suited to the present work:

⋆ When $n=1$ this is simply NEWTON's method for finding the zeros of a polynomial of degree l.

Theorem 1. *For a simple λ-matrix $D_l(\lambda)$ we have*

$$[D_l(\lambda)]^{-1} = Q(I\lambda - \Lambda)^{-1} R', \tag{19}$$

provided $\lambda \neq \lambda_i$, and provided that the latent vectors are normalised so that

$$\boldsymbol{r}_i' D_l^{(1)}(\lambda_i)\, \boldsymbol{q}_i = 1. \tag{20}$$

From our definition we can write Q as the partitioned matrix:

$$[\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_{ln}],$$

and similarly for R; while $(I\lambda - \Lambda)^{-1}$ has diagonal elements $(\lambda - \lambda_1)^{-1}, \dots, (\lambda - \lambda_{ln})^{-1}$. Using these forms on the right of (19) and finding the product of the partitioned matrices, we see that (19) can also be written as:

$$[D_l(\lambda)]^{-1} = \sum_{i=1}^{ln} \frac{F_i}{\lambda - \lambda_i}, \tag{21}$$

where the matrices F_i are the outer products $\boldsymbol{q}_i \boldsymbol{r}_i'$. Notice that (19) and (21) are in fact the same equation written in two different ways, and either of these forms may be used under the conditions of theorem 1. In particular, *multiple* roots are permitted in the partial fraction expansion (21), provided the degeneracy condition in our definition of a simple λ-matrix is satisfied (*cf.* Frazer, Duncan & Collar [7]).

7. The approximate latent vectors

If the sequence λ_s tends towards a latent root, λ, which is a simple zero of $|D_l(\lambda)|$, then the sequences of vectors $\boldsymbol{\xi}_s$ and $\boldsymbol{\eta}_s$ defined by (16) tend towards the (unnormalised) latent vectors $\boldsymbol{q}$ and $\boldsymbol{r}$. This is apparent if we write $[D_l(\lambda_s)]^{-1}$ in the form (21). We have

$$\boldsymbol{\xi}_s = [D_l(\lambda_s)]^{-1} \boldsymbol{w} = \sum_{i=1}^{ln} \frac{F_i \boldsymbol{w}}{\lambda_s - \lambda_i}. \tag{22}$$

Now $F_i \boldsymbol{w} = (\boldsymbol{q}_i \boldsymbol{r}_i') \boldsymbol{w} = \boldsymbol{q}_i (\boldsymbol{r}_i' \boldsymbol{w}) = \boldsymbol{q}_i \omega_i$ say, where

$$\omega_i = \boldsymbol{r}_i' \boldsymbol{w}. \tag{23}$$

If we denote the latent root to which the λ_s converge by λ, the corresponding vectors by $\boldsymbol{q}$ and $\boldsymbol{r}$, and put $\boldsymbol{r}'\boldsymbol{w} = \omega$, then we may combine (22) and (23) to give

$$\boldsymbol{\xi}_s = \frac{\omega}{\lambda_s - \lambda} \boldsymbol{q} + \sum_{i=1}^{ln}{}' \frac{\omega_i}{\lambda_s - \lambda_i} \boldsymbol{q}_i, \tag{24}$$

where $\sum'$ implies a summation in which the approximated latent root is omitted. Thus provided λ_s is sufficiently close to λ and the first of the inequalities (18) is satisfied, the dominant part of $\boldsymbol{\xi}_s$ is proportional to $\boldsymbol{q}$.

Similarly, we obtain (for a simple zero, λ)

$$\boldsymbol{\eta}_s = \frac{\zeta}{\lambda_s - \lambda} \boldsymbol{r} + \sum_{i=1}^{ln}{}' \frac{\zeta_i}{\lambda_s - \lambda_i} \boldsymbol{r}_i, \tag{25}$$

where

$$\zeta_i = \boldsymbol{q}_i' \boldsymbol{z}. \tag{26}$$

If the latent root has multiplicity α, and $1 \leqq \alpha \leqq n$, then the α latent vectors, $\boldsymbol{q}_\nu$ say, corresponding to λ are linearly independent (for a *simple* λ-matrix). In this case the vector $\omega \boldsymbol{q}$ appearing in (24) must be replaced by $\sum \omega_\nu \boldsymbol{q}_\nu$, and $\boldsymbol{\xi}_s$ will tend towards a linear combination of the α vectors $\boldsymbol{q}_\nu$. By taking α independent vectors $\boldsymbol{w}$, the various limits correspond to α linearly independent combinations of the $\boldsymbol{q}_\nu$, and the sub-space generated by the latent vectors of λ is then completely determined. The vectors themselves are not determined to within an arbitrary orthogonal transformation, after being normalised according to (20).

In order to verify this last statement, we note first that any linear combination of the $\boldsymbol{q}_\nu$'s (or of the $\boldsymbol{r}_\nu$'s) is a latent vector with latent root λ. Next we note that the sets of latent vectors $\boldsymbol{q}_\nu$ and $\boldsymbol{r}_\mu$ (associated with λ), after being normalised according to (20), satisfy the following condition:

$$\boldsymbol{r}'_\mu D_l^{(1)}(\lambda)\, \boldsymbol{q}_\nu = \delta_{\mu\nu},$$

where we have $1 \leqq \mu,\ \nu \leqq \alpha$. (This is easily deduced from equation (20) of [3].) If we now denote by Q_α and R_α the $n \times \alpha$ matrices whose columns are the vectors $\boldsymbol{q}_\nu$ and $\boldsymbol{r}_\nu$ respectively, then the above condition can be written in the form

$$R'_\alpha D_l^{(1)}(\lambda)\, Q_\alpha = I_\alpha .$$

Let G be an arbitrary $\alpha \times \alpha$ orthogonal matrix, and replace Q_α and R_α in the previous equation by $Q_\alpha G$ and $R_\alpha G$; then

$$\begin{aligned}(R_\alpha G)' D_l^{(1)}(\lambda)\,(Q_\alpha G) &= G' R'_\alpha D_l^{(1)}(\lambda)\, Q_\alpha G\\ &= G' I_\alpha G\\ &= I_\alpha .\end{aligned}$$

Thus the component vectors of $Q_\alpha G$ and $R_\alpha G$ satisfy the same conditions as those of Q_α and R_α. That is, these vectors are not defined within an arbitrary orthogonal transformation.

Another argument justifying the use of (16) has been given by OSTROWSKI [8].

8. Convergence

In this section we establish a "local" result concerning the convergence of the iterative scheme described in Section 5 — when applied to simple λ-matrices. First we prove the following lemmas.

Lemma 1. *Under the conditions of theorem* 1 *we have*

$$\frac{d}{d\lambda} D_l(\lambda) = D_l(\lambda)\, Q\,(I\lambda - \Lambda)^{-2} R' D_l(\lambda) . \tag{27}$$

For any square matrix U of order n whose elements are differentiable functions of a scalar, λ, and for which U^{-1} exists, we have

$$U(\lambda)\, U^{-1}(\lambda) = I,$$

so that

$$\left[\frac{d}{d\lambda} U(\lambda)\right] U^{-1}(\lambda) = -\, U(\lambda) \frac{d}{d\lambda} U^{-1}(\lambda),$$

and
$$\frac{d}{d\lambda}U(\lambda) = -U(\lambda)\left[\frac{d}{d\lambda}U^{-1}(\lambda)\right]U(\lambda).$$

Putting $U = D_l$ and making use of (19) gives the result at once.

Lemma 2. *Under the conditions of theorem* 1 *the sequence* λ_s, $s = 0, 1, 2, \ldots,$ *defined by* (16) *and* (17) *is given by*

$$\lambda_{s+1} = \frac{\boldsymbol{z}'\, Q\, \Lambda (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w}}{\boldsymbol{z}'\, Q\, (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w}}. \tag{28}$$

Using (16) we obtain for the numerator of the quotient in (17):

$$\begin{aligned} \boldsymbol{\eta}_s'\, D_l(\lambda_s)\, \boldsymbol{\xi}_s &= \boldsymbol{z}'[D_l(\lambda_s)]^{\,1}\, \boldsymbol{w}, \\ &= \boldsymbol{z}'\, Q\, (I\,\lambda_s - \Lambda)^{-1} R'\, \boldsymbol{w}, \end{aligned} \tag{29}$$

using (19). For the denominator of the quotient in (17) we obtain from (16) and (27):

$$\boldsymbol{\eta}_s'\, D_l^{(1)}(\lambda_s)\, \boldsymbol{\xi}_s = \boldsymbol{z}'\, Q\, (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w}.$$

This can be combined with (29) to give

$$\lambda_s\, \boldsymbol{\eta}_s'\, D_l^{(1)}(\lambda_s)\, \boldsymbol{\xi}_s - \boldsymbol{\eta}_s'\, D_l(\lambda_s)\, \boldsymbol{\xi}_s = \boldsymbol{z}'\, Q\Lambda (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w};$$

and substitution in (17) now gives the desired result.

Theorem 2. *If the conditions of theorem* 1 *apply and the sequence* λ_s *converges to a latent root, then the convergence is at least quadratic.*

If the sequence λ_s converges to the latent root λ, then using lemma 2 we may write

$$\lambda_{s+1} - \lambda = \frac{\boldsymbol{z}'\, Q\, (\Lambda - \lambda I)\, (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w}}{\boldsymbol{z}'\, Q\, (I\,\lambda_s - \Lambda)^{-2} R'\, \boldsymbol{w}}. \tag{30}$$

From (23) and (26) we see that the $l\,n \times 1$ vectors $R'\boldsymbol{w}$ and $Q'\boldsymbol{z}$ are conveniently denoted by $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$, these vectors having elements ω_i, ζ_i respectively, for $i = 1, 2, \ldots, l\,n$. Thus

$$\lambda_{s+1} - \lambda = \frac{\boldsymbol{\zeta}'(\Lambda - \lambda I)\, (I\,\lambda_s - \Lambda)^{-2}\, \boldsymbol{\omega}}{\boldsymbol{\zeta}'(I\,\lambda_s - \Lambda)^{-2}\, \boldsymbol{\omega}} = \frac{\displaystyle\sum_{i=1}^{l\,n} \zeta_i\, \omega_i \frac{\lambda_i - \lambda}{(\lambda_s - \lambda_i)^2}}{\displaystyle\sum_{i=1}^{l\,n} \frac{\zeta_i\, \omega_i}{(\lambda_s - \lambda_i)^2}}, \tag{31}$$

since the square matrices in the first expression are all diagonal matrices. In the final expression on the right of (31) any λ_i for which all the corresponding products $\zeta_i\, \omega_i$ vanish does not appear in either summation. Denote the *distinct* λ_i for which at least one product $\zeta_i\, \omega_i$ is non-zero by $\sigma_1, \sigma_2, \ldots, \sigma_m$. Then (31) can be written:

$$\lambda_{s+1} - \lambda = \frac{\displaystyle\sum_{i=1}^{m} p_i \frac{\sigma_i - \lambda}{(\lambda_s - \sigma_i)^2}}{\displaystyle\sum_{i=1}^{m} \frac{p_i}{(\lambda_s - \sigma_i)^2}},$$

where the p_i are non-zero constants. If we denote by p the value of p_i corresponding to the latent root λ ($p \neq 0$ by virtue of (18)), we have

$$\begin{aligned}\lambda_{s+1}-\lambda &= \frac{\sum' p_i \dfrac{\sigma_i-\lambda}{(\lambda_s-\sigma_i)^2}}{\dfrac{p}{(\lambda_s-\lambda)^2}+\dfrac{1}{(\lambda_s-\lambda)^2}\sum' p_i \dfrac{(\lambda_s-\lambda)^2}{(\lambda_s-\sigma_i)^2}} \\ &= (\lambda_s-\lambda)^2 \frac{\sum' p_i \dfrac{\sigma_i-\lambda}{(\lambda_s-\sigma_i)^2}}{p+\sum' p_i \dfrac{(\lambda_s-\lambda)^2}{(\lambda_s-\sigma_i)^2}},\end{aligned} \tag{32}$$

where $\sum'$ now implies a summation from 1 to m in which the approximated latent root is omitted. Thus, if $\lambda_s \to \lambda$, we find that

$$\frac{\lambda_{s+1}-\lambda}{(\lambda_s-\lambda)^2} \to \frac{1}{p}\sum' \frac{p_i}{\sigma_i-\lambda}, \tag{33}$$

and hence the convergence will in general be *quadratic*.

9. The convergence neighbourhood

We now generalise the treatment of OSTROWSKI [2_1] to establish a convergence neighbourhood for the above process. That is, to define limits within which an approximation λ_s (to λ) must lie in order that $|\lambda_{s+1}-\lambda| < |\lambda_s-\lambda|$, and the successive approximations to λ are such that the sequence $|\lambda_s-\lambda|$ is strictly decreasing to zero as s increases.

Let

$$d = \operatorname{Min}|\sigma_i-\lambda|,$$

and suppose that N ($\geqq 2$) is a constant such that

$$|\lambda_s-\lambda| \leqq \frac{d}{N}, \quad \text{and hence} \quad |\lambda_s-\sigma_i| \geqq d\left(1-\frac{1}{N}\right).$$

Then

$$\left|\frac{\sigma_i-\lambda}{(\lambda_s-\sigma_i)^2}\right| \leqq \frac{1}{d}\,\frac{N^2}{(N-1)^2}, \quad \text{and} \quad \left|\frac{(\lambda_s-\lambda)^2}{(\lambda_s-\sigma_i)^2}\right| \leqq \frac{1}{(N-1)^2}. \tag{34}$$

Using the inequality $|\underline{a}+\underline{b}| \geqq \big||\underline{a}|-|\underline{b}|\big|$ for any complex $\underline{a}$ and $\underline{b}$, we obtain from (32);

$$\left|\frac{\lambda_{s+1}-\lambda}{(\lambda_s-\lambda)^2}\right| \leqq \frac{\left|\sum' p_i \dfrac{\sigma_i-\lambda}{(\lambda_s-\sigma_i)^2}\right|}{\left||p|-\left|\sum' p_i \dfrac{(\lambda_s-\lambda)^2}{(\lambda_s-\sigma_i)^2}\right|\right|},$$

and we assume that the denominator does not vanish.

Put $P=\sum_{i=1}^{m}|p_i|$ and $\pi=|p|$, so that $\sum'|p_i| = P-\pi$. Then using the inequalities (34) we get

$$\left|\frac{\lambda_{s+1}-\lambda}{(\lambda_s-\lambda)^2}\right| \leqq \frac{1}{d}\,\frac{N^2(P-\pi)/(N-1)^2}{\pi-(P-\pi)/(N-1)^2}; \tag{35}$$

a sufficient condition for the manipulation of the denominator being that $\pi > (P-\pi)/(N-1)^2$, *i.e.*

$$\pi > \frac{1}{N^2-2N+2}\,P. \tag{36}$$

Under this condition (35) can be written

$$\left|\frac{\lambda_{s+1}-\lambda}{\lambda_s-\lambda}\right| \leqq \frac{N}{d}\,\frac{N(P-\pi)}{(N^2-2N+2)\,\pi-P}\,|\lambda_s-\lambda|, \tag{37}$$

and we deduce that:

If for some s we have

$$|\lambda_s-\lambda| < \frac{d}{N}\,\mathrm{Min}\left\{\frac{(N^2-2N+2)\,\pi-P}{N(P-\pi)},\,1\right\}, \tag{38}$$

and (36) *is satisfied, then from this index onwards the distances* $|\lambda_s-\lambda|$ *are strictly diminishing and converge to zero.*

The strength of this result will depend on l. For $l=1$ and all the latent roots real and positive OSTROWSKI [2_1] uses $N=2$ to obtain a comparatively large convergence neighbourhood. If we allow complex roots then (38) gives a convergence neighbourhood of radius $d/2$ provided $\pi>3P/4$. In the case examined by OSTROWSKI the same convergence neighbourhood is obtained if $\pi>2P/3$; a slightly stronger result.

Notice that in the case $l=1$ it is always *possible* that $\pi=P$. That is, that $p_i=0$ for $p_i\neq p$. This is achieved if, for each latent root $\lambda_i\neq\lambda$, either $\omega_i=0$ or $\zeta_i=0$ (*re*: equations (23) and (26)).

However, if $l>1$ there is an important class of matrices for which we have the strict inequality $\pi<P$; namely those for which all sets of right latent vectors taken n at a time are linearly independent and similarly for the left latent vectors. The proof of this is easy, but we will not give it here. From the assumption (18) we observe that $p\neq0$ so that for this class of λ-matrices there must be at least one other coefficient, p_i, which is non-zero, and for large l there may be a lot more than one. Thus for $l>1$ the condition $\pi>3P/4$ obtained from (38) by taking $N=2$ may be quite a stringent one and, in theory at least, a smaller convergence neighbourhood may be more reasonable. For instance, by taking $N=4$ we obtain a convergence neighbourhood of radius $d/4$ provided $\pi>5P/14\simeq(0{\cdot}36)P$.

Hence it appears that, for given starting vectors $\boldsymbol{w}$ and $\boldsymbol{z}$, the accuracy of the initial guess λ_0 will have to increase with l, the degree of the λ-matrix; while $\boldsymbol{w}$ and $\boldsymbol{z}$ should be chosen to make π $\left(=|p|=|\boldsymbol{q}'\boldsymbol{z}\cdot\boldsymbol{r}'\boldsymbol{w}|\right.$ for a simple root$\left.\right)$ approach P $\left(=\sum|p_i|\right)$ as nearly as possible.

10. Broken iteration

A second iterative process known as broken iteration is considered by OSTROWSKI [2_3] and by CRANDALL [4], and suggests the following modified process:

For any λ_s *define*

$$\boldsymbol{\xi}_s=[D_l(\lambda_s)]^{-1}\boldsymbol{\xi}_{s-1},\qquad \boldsymbol{\eta}_s=[D_l'(\lambda_s)]^{-1}\boldsymbol{\eta}_{s-1} \tag{39}$$

for $s=0,1,2,\ldots,$ *where* $\boldsymbol{\xi}_{-1}$ *and* $\boldsymbol{\eta}_{-1}$ *are arbitrary starting vectors* $\boldsymbol{w}$ *and* $\boldsymbol{z}$ *satisfying* (18); *and put*

$$\lambda_{s+1}=R(\boldsymbol{\xi}_s,\boldsymbol{\eta}_s,\lambda_s). \tag{40}$$

This scheme should be compared with that of Section 5. The arbitrary vectors $\boldsymbol{w}$ and $\boldsymbol{z}$ used at *each step* of the former scheme are now replaced by the successive approximations $\boldsymbol{\xi}_{s-1}$, $\boldsymbol{\eta}_{s-1}$ to the latent vectors $\boldsymbol{q}$ and $\boldsymbol{r}$. When $l=1$ it can be shown that if the above scheme is slightly modified to take advantage of the orthogonality properties of the latent vectors (*cf.* CRANDALL [*4*]), then *cubic* convergence may be achieved. However, it appears that if $l>1$ this rate of convergence cannot be achieved, at least for the class of matrices defined at the end of the previous section (*i.e.* those for which $\pi<P$). Using broken iteration may increase the rate of convergence, but probably not by another order of magnitude.

11. Numerical examples

a) The oscillations of a wing in an airstream

This example is taken from the work of FRAZER, DUNCAN & COLLAR [*9*]. With one or two obvious modifications to the numerical values the problem reduces to finding the latent roots and vectors of $D_2(\lambda)$, where

$$A_0=\begin{bmatrix} 17{\cdot}6 & 1{\cdot}28 & 2{\cdot}89 \\ 1{\cdot}28 & 0{\cdot}824 & 0{\cdot}413 \\ 2{\cdot}89 & 0{\cdot}413 & 0{\cdot}725 \end{bmatrix}, \qquad A_1=\begin{bmatrix} 7{\cdot}66 & 2{\cdot}45 & 2{\cdot}10 \\ 0{\cdot}230 & 1{\cdot}04 & 0{\cdot}223 \\ 0{\cdot}600 & 0{\cdot}756 & 0{\cdot}658 \end{bmatrix},$$

$$A_2=\begin{bmatrix} 121 & 18{\cdot}9 & 15{\cdot}9 \\ 0 & 2{\cdot}70 & 0{\cdot}145 \\ 11{\cdot}9 & 3{\cdot}64 & 15{\cdot}5 \end{bmatrix}.$$

From experience of similar problems a person experienced in aeroelastic work may take as first approximations to the largest latent root and associated right latent vector:

$$\lambda_0=i(10{\cdot}0) \quad \text{and} \quad \boldsymbol{w}'=[0 \;\; 0 \;\; 1{\cdot}0].$$

For convenience we take $\boldsymbol{z}=\boldsymbol{w}$ in equations (16). After inverting the matrix $D_2(\lambda_0)$ we find from (16) that

$$\boldsymbol{\xi}_0=\begin{bmatrix} -0{\cdot}1481+i(0{\cdot}00289) \\ -0{\cdot}2764+i(0{\cdot}01700) \\ 1{\cdot}0 \end{bmatrix}, \qquad \boldsymbol{\eta}_0=\begin{bmatrix} -0{\cdot}1476-i(0{\cdot}00528) \\ -0.2716+i(0{\cdot}02126) \\ 1{\cdot}0 \end{bmatrix},$$

where, to facilitate comparisons, we normalise the vectors so that the largest element is one. Substituting λ_0, $\boldsymbol{\xi}_0$ and $\boldsymbol{\eta}_0$ in (17) (and using the fact that $\boldsymbol{\eta}_s' D_l(\lambda_s)\,\boldsymbol{\xi}_s=\boldsymbol{w}'\boldsymbol{\xi}_s$) it is found that

$$\lambda_1=-0{\cdot}756+i(8{\cdot}535).$$

This completes one cycle of the iteration. In this case the choice of $\boldsymbol{w}$ is not decisive in determining the question of convergence. It is found that with the above choice of λ_0 and $\boldsymbol{w}'=[1{\cdot}0 \;\; 0 \;\; 0]$ or $\boldsymbol{w}'=[0 \;\; 1{\cdot}0 \;\; 0]$ the process would still converge, and hence would do so for an arbitrary choice of $\boldsymbol{w}$.

In the next cycle of operations it is found that

$$\boldsymbol{\xi}_1 = \begin{bmatrix} -0{\cdot}14938 + i(0{\cdot}004416) \\ -0{\cdot}27756 - i(0{\cdot}020362) \\ 1{\cdot}0 \end{bmatrix}, \qquad \boldsymbol{\eta}_1 = \begin{bmatrix} -0{\cdot}14663 - i(0{\cdot}005382) \\ -0{\cdot}26753 - i(0{\cdot}023272) \\ 1{\cdot}0 \end{bmatrix}$$

and then

$$\lambda_2 = -0{\cdot}8834 + i(8{\cdot}4411).$$

To this accuracy the latent root is in fact

$$-0{\cdot}8848 + i(8{\cdot}4415).$$

The value given in [*9*] is $-0{\cdot}885 + i(8{\cdot}443)$, so the fourth significant figure of the calculations quoted there is suspect. Bearing this in mind, the corresponding right latent vector as given in [*9*] is (after renormalisation)

$$\begin{bmatrix} -0{\cdot}1494 + i(0{\cdot}00466) \\ -0{\cdot}2778 - i(0{\cdot}02066) \\ 1{\cdot}0 \end{bmatrix}.$$

We observe that in this case results which are sufficiently accurate for most engineering purposes are obtained after only two steps of the iteration. A further important feature of the method is the possibility of finding latent roots in any order. This is in contrast to the iterative (power) method used in [*9*] which first gives the dominant root, and the accuracy with which sub-dominant roots are calculated depends on the accuracy of those already found.

However, the example chosen above may not be typical of problems met in practice, as the distribution of the latent roots is particularly favourable. The complete set of latent roots is:

$$-0{\cdot}885 \pm i(8{\cdot}442), \qquad 0{\cdot}097 \pm i(2{\cdot}522), \qquad -0{\cdot}920 \pm i(1{\cdot}760).$$

b) Perturbation problems

We consider here a rather special perturbation problem. A mechanical system of n degrees of freedom oscillating about a position of equilibrium in the presence of small viscous damping forces gives rise to the following equations:

$$A\ddot{\boldsymbol{p}} + \varepsilon B\dot{\boldsymbol{p}} + C\boldsymbol{p} = 0,$$

where A, B and C are $n \times n$ real symmetric matrices of constants, ε is a scalar which is small compared to the elements of A, B and C, and $\boldsymbol{p}$ is the column vector of generalised coordinates. Thus the term $\varepsilon B\dot{\boldsymbol{p}}$ represents a perturbation from the conservative system obtained when $\varepsilon = 0$. On looking for a solution in the form $\boldsymbol{p} = \boldsymbol{q}e^{\lambda t}$, we obtain

$$(A\lambda^2 + \varepsilon B\lambda + C)\boldsymbol{q} = 0. \tag{41}$$

We take as our first approximation to a solution of (41) a solution obtained when $\varepsilon = 0$, say λ_0, $\boldsymbol{q}_0$. It is well known that $\boldsymbol{q}_0$ may be normalised so that

$$\boldsymbol{q}_0' A\boldsymbol{q}_0 = 1, \quad \text{and} \quad \boldsymbol{q}_0' C\boldsymbol{q}_0 = -\lambda_0^2 \tag{42}$$

(*cf.* equations (7)). In order to improve on the approximation λ_0 we use (17) with $\boldsymbol{w}=\boldsymbol{z}=\boldsymbol{q}_0$ and find

$$\lambda_1=\lambda_0-\frac{\boldsymbol{q}_0'(A\,\lambda_0^2+\varepsilon B\,\lambda_0+C)\,\boldsymbol{q}_0}{\boldsymbol{q}_0'(2A\,\lambda_0+\varepsilon B)\,\boldsymbol{q}_0}.$$

Using (42) this reduces to

$$\lambda_1=\lambda_0-\frac{\varepsilon\beta\,\lambda_0}{2\,\lambda_0+\varepsilon\,\beta},$$

where $\beta=\boldsymbol{q}_0'B\,\boldsymbol{q}_0$. If we assume that the exact latent vector $\boldsymbol{q}$ differs from $\boldsymbol{q}_0$ by terms of order ε, then we have

$$\lambda=\lambda_0-\tfrac{1}{2}\varepsilon\beta+O(\varepsilon^2),$$

or, to the order of ε indicated,

$$\lambda_1=\lambda_0-\tfrac{1}{2}\varepsilon\beta. \tag{43}$$

This first correction was obtained by Lord RAYLEIGH [*10*], and has been discussed further by the author [*11*]. As a numerical example we consider that of Section 5 of [*11*]. The matrices appearing in (41) (after transformation to *suitable* principal coordinates for the undamped system) are:

$$A=I_3,\quad B=\begin{bmatrix}0{\cdot}602856 & 0 & -0{\cdot}255274\\ & 0{\cdot}966840 & -0{\cdot}216403\\ & & 5{\cdot}761640\end{bmatrix},\quad C=\begin{bmatrix}1{\cdot}0 & 0 & 0\\ 0 & 1{\cdot}0 & 0\\ 0 & 0 & 2{\cdot}0\end{bmatrix};$$

and we take $\varepsilon=0{\cdot}01$.

In (16) we have $\boldsymbol{w}=\boldsymbol{z}=\boldsymbol{q}_0$, so that the sequences $\boldsymbol{\xi}_s$ and $\boldsymbol{\eta}_s$ coincide. For λ_0 we take the isolated root of the undamped system, *i.e.* $\lambda_0=i\sqrt{2}$. Clearly then, $\boldsymbol{q}_0$ is the column vector with elements 0, 0, 1. We alter the order of operations in (16) and (17) and write

$$\boldsymbol{\xi}_1=[D_2(\lambda_0)]^{-1}\,\boldsymbol{q}_0=[A\,\lambda_0^2+\varepsilon\,B\,\lambda_0+C]^{-1}\,\boldsymbol{q}_0.$$

Now from (23) we have in this case $\omega_i=\boldsymbol{q}_i'\boldsymbol{q}_0$, and since $\boldsymbol{q}_0$ differs from $\boldsymbol{q}$ by terms of order ε, the orthogonality property shows that $\omega_i=O(\varepsilon)$. From (24) we now see that after being normalised, $\boldsymbol{\xi}_1=\boldsymbol{q}+O(\varepsilon^2)$. Calculation gives

$$\boldsymbol{\xi}_1=\begin{bmatrix}0{\cdot}000031-i(0{\cdot}003610)\\ 0{\cdot}000042-i(0{\cdot}003060)\\ 1{\cdot}0\end{bmatrix},$$

and this vector has been normalised to make its largest element one. (This manipulation will obviously not affect (17).) To the same order of accuracy we get immediately from (43):

$$\begin{aligned}\lambda_1&=i\sqrt{2}-(0{\cdot}005)\,(5{\cdot}761640)\\ &=-0{\cdot}0288082+i(1{\cdot}4142136).\end{aligned}$$

Then the second application of (17) gives

$$\begin{aligned}\lambda_2&=R(\boldsymbol{\xi}_1,\lambda_1)\\ &=-0{\cdot}0288086+i(1{\cdot}4139122),\end{aligned}$$

and the error should be $O(\varepsilon^4)$. In fact the exact root and vector are:

$$\lambda = -0{\cdot}0288084 + i(1{\cdot}4139127), \quad \boldsymbol{q} = \begin{bmatrix} -0{\cdot}000189 - i(0{\cdot}003601) \\ -0{\cdot}000145 - i(0{\cdot}003054) \\ 1{\cdot}0 \end{bmatrix}.$$

The method used in this example may obviously be used to investigate the effect of a small perturbation on any known latent root and vector of a lambda matrix — the first order perturbation of the latent root being particularly easy to obtain.

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Elementary Function Representations of Schlömilch Series

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1. Introduction

Alternative representations, as series of more elementary functions, are obtained for the Schlömilch series[1]

$$\sum_{s=1}^{\infty} Z_{2n}(sD)\cos(sD\sin\vartheta_0), \tag{1}$$

$$\sum_{s=1}^{\infty} Z_{2n+1}(sD)\sin(sD\sin\vartheta_0), \tag{2}$$

where Z_n is any solution of Bessel's equation for integral n, and where $D>0$, $0\leqq\sin\vartheta_0<1$. The series converge provided $D(1+\sin\vartheta_0)/2\pi$ and $D(1-\sin\vartheta_0)/2\pi$ do not equal integers.

These series are of interest in a variety of physical problems. In particular the special case of (1) with $\vartheta_0=0$, which was considered by IGNATOWSKY [*2*], arises in the analysis of the scattering of a plane wave normally incident on a diffraction grating [*3*] (*i.e.*, on a planar periodic array of parallel cylinders). For such problems, we have $D=kd=2\pi d/\lambda$, where d is the spacing between cylinders and λ is the wavelength. The more general series (1) and (2) arise for arbitrary angle of incidence on the grating [*4*], for related waveguide problems, *etc.* Various special cases of (1) and (2) arise in other scattering problems, in characteristic value problems in the theory of vibrations, and in communications engineering.

The series we are concerned with may be collected in the form

$$\mathscr{H}_n=\sum_{s=1}^{\infty} H_n(sD)\left[e^{isD\sin\vartheta_0}(-1)^n+e^{-isD\sin\vartheta_0}\right], \tag{3}$$

where $H_n=H_n^{(1)}=J_n+iN_n$ represents the usual Hankel function of the first kind, and where J_n and N_n are the Bessel and Neumann functions. In general[2], the representation (3) is too slowly convergent either for numerical computations or for the determination of the explicit dependence of $\mathscr{H}_n$ on the parameters n, D, and ϑ_0. The purpose of this paper is to derive an alternative representation in terms of elementary functions. The case $n=0$ is given by MAGNUS & OBERHETTINGER [*6*] (pp. 59, 60) and will not be evaluated.

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[1] See WATSON [*1*], Ch. XIX, for a general discussion of such series.

[2] For some purposes adequate results may be obtained from a representation of (3) based on the asymptotic form of H_n for $D\gg n$, $D\gg 1$. See [*5*].

In order to obtain a more useful form, we consider initially

$$
\begin{aligned}
h_n(R) &= \sum_{s=-\infty}^{\infty} e^{isD\sin\vartheta_0} H_n(R_s)\, i^n e^{-in\varphi_s} - H_n(R_0)\, i^n e^{-in\varphi_0}, \\
R_s^2 &= X^2 + (Y - sD)^2, \quad R_0 = R = (X^2 + Y^2)^{\frac{1}{2}}, \quad \varphi_s = \tan^{-1}\left(\frac{Y - sD}{X}\right).
\end{aligned}
\tag{4}
$$

As $R \to 0$, we see that $R_s \to |sD|$, and $\varphi_s \to \pm\frac{1}{2}\pi$ for $s \lessgtr 0$; consequently we may express (3) as

$$
\mathscr{H}_n = \lim_{R\to 0} h_n(R). \tag{5}
$$

In the above, we have essentially added and subtracted the term containing $H_n(R)$, there being no singular terms in the limit. This step was introduced by IGNATOWSKY [2] for $\vartheta_0 = 0$; aside from this, our procedure is quite different from IGNATOWSKY'S and obviates his detailed algebraic manipulations: Instead of using series expansions of $H_n(R)$ for $R \to 0$, we recast h_n as the difference of a sum minus the analogous integral, *i.e.*, $\left(\sum_{\mu} - \int d\mu\right) f(\mu)$. The limit of this result is then split into two parts, one which is rapidly convergent, and one which by the use of Euler's summation formula can be written as a finite sum.

It will be seen that the parameter Y is unessential in the above and could be set equal to zero from the start; however, its inclusion provides a slight generalization of interest to related physical problems. For such problems we have $X = kx$ and $Y = ky$, where x and y are the usual Cartesian coordinates.

2. Derivation of an Operational Form

Using the Sommerfeld integral representation of the Hankel function [6, p. 41], and exploiting the full range available for the limits of the path in the complex domain, we write

$$
H_n(R_s)\, i^n e^{-in\varphi_s} = \frac{1}{\pi} \int_{-\frac{1}{2}\pi + i\infty}^{\frac{1}{2}\pi - i\infty} e^{iR_s\cos(\tau+\varphi_s) + in\tau}\, d\tau, \tag{6}
$$

which holds for all $|\varphi_s| < \frac{1}{2}\pi$, *i.e.*, $X > 0$. (Note that the limits are the same for all s in spite of the fact that the φ_s differ.) Equivalently, in terms of $t = \sin\tau$, we have

$$
H_n(R_s)\, i^n e^{-in\varphi_s} = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i(Y-sD)t + iX\sqrt{1-t^2}}\, \frac{e^{in\sin^{-1}t}}{\sqrt{1-t^2}}\, dt, \tag{7}
$$

where $\sqrt{1-t^2} = i\sqrt{t^2-1}$ if $|t| > 1$. (The case $n = 0$ is given in [6, p. 40].)

We now substitute (7) into (4), and use POISSON'S summation formula in the form $\frac{1}{2\pi} \sum_{s=-\infty}^{\infty} e^{isa} \int_{-\infty}^{\infty} f(t)\, e^{ist}\, dt = \sum_{\mu=-\infty}^{\infty} f(2\mu\pi + a)$ as in [6, p. 217] (or, equivalently, identify the sum over s as the "periodic δ-function"). Thus, in terms of

$$
\begin{gathered}
\sin\vartheta_\mu \equiv \sin\vartheta_0 + 2\mu\pi/D, \qquad \mu = 0, \pm 1, \ldots, \\
C_\mu = 1/D\sqrt{1 - \sin^2\vartheta_\mu} \text{ if } \sin^2\vartheta_\mu < 1, \quad C_\mu = 1/iD\sqrt{\sin^2\vartheta_\mu - 1} \text{ if } \sin^2\vartheta_\mu > 1,
\end{gathered}
\tag{8}
$$

we obtain

$$(9)\qquad \sum_{s=-\infty}^{\infty} e^{isD\sin\vartheta_0} H_n(R_s)\, i^n\, e^{-in\varphi_s} = 2\sum_{\mu=-\infty}^{\infty} C_\mu\, e^{-in\vartheta_\mu + iY\sin\vartheta_\mu + iX\cos\vartheta_\mu}, \qquad X>0.$$

Similarly for $X<0$, we replace φ_s by $\pi-\varphi_s$ in (6), and obtain (9) with ϑ_μ replaced by $\pi-\vartheta_\mu$.

Alternative derivations of (9) are given by REICHE [*7*] and IGNATOWSKY [*2, 3*]. (The present derivation is briefer, and provides results and notation we require subsequently.) The special case $n=0$ is also a special case of an expression on p. 62 of MAGNUS & OBERHETTINGER [*6*] (and the result for $n\neq 0$ may also be obtained, after some manipulation, from their related expression). Related series are considered by SCHMIDT *et al.* [*8*], who also give additional references.

The series in (9) converge provided that

$$D(1\pm\sin\vartheta_0)/2\pi \equiv m_\mp \neq 0, 1, 2, \ldots.$$

Were m_+ or m_- an integer, then the corresponding value of C_μ in the right-hand side of (9) would be infinite. Similarly, since $H_n(kR_s)\sim i^{n}\sqrt{\frac{2}{\pi|s|D}}\; e^{i|s|D-i\pi/4}$, the left side of (9) would also diverge for integral m_+ or m_-. (For non-integral values, convergence considerations are obviated by using $R_s(1+i\varepsilon)$, $\varepsilon\to+0$.)

Using the terminology of grating theory [*4*], we may distinguish two kinds of terms, or "modes", in the sum over μ. (All the angles are measured from the grating's normal, *i.e.*, $|\mathrm{Re}\,\vartheta_\mu|\leq\frac{1}{2}\pi$.) The "propagating modes" correspond to $|\sin\vartheta_\mu|<1$, $\cos\vartheta_\mu=\sqrt{1-\sin^2\vartheta_\mu}$; for these, we have $-\mu_-\leq\mu\leq\mu_+$ where μ_+ and μ_- are the closest integers to m_+ and m_- for which $|\sin\vartheta_{\mu\pm}|<1$, *i.e.*, $\mu_+=[m_+]=[D(1-\sin\vartheta_0)/2\pi]$, and $\mu_-=[m_-]=[D(1+\sin\vartheta_0)/2\pi]$. The remaining values $\mu>\mu_++1>m_+$, and $\mu<-\mu_--1<-m_-$ are "evanescent modes" for which $|\sin\vartheta_\mu|>1$, $\cos\vartheta_\mu=i\sqrt{\sin^2\vartheta_\mu-1}$; for these we may use $\cosh\eta_\mu=\sin\vartheta_0+2\mu\pi/D$, where $\vartheta_\mu=\frac{1}{2}\pi-i|\eta_\mu|$ for $\mu>\mu_++1$, and $\vartheta_\mu=-\frac{1}{2}\pi+i|\eta_\mu|$ for $\mu<-\mu_--1$. (The excluded values $\sin\vartheta_\mu=\pm1$, which correspond to integral values of $m_\pm=D(1\mp\sin\vartheta_0)/2\pi$, are known as the "grazing modes" or "Rayleigh values".)

Returning to (4), we use the integral representation as in (7) for the isolated term $H_n(R_0)\, i^n\, e^{-in\varphi_0}$. For convenience, we write

$$(10)\qquad H_n(R_0)\, i^n\, e^{-in\varphi_0} = 2\int_{-\infty}^{\infty} C_\mu\, e^{-in\vartheta_\mu+iY\sin\vartheta_\mu+iX\cos\vartheta_\mu}\, d\mu, \qquad X>0,$$

where (in order to obtain the same kernel as in (9)) we changed the variable of integration to μ such that $-t=\sin\vartheta_0+2\pi\mu/D=\sin\vartheta_\mu$.

Thus, for $X>0$, we obtain

$$(11)\qquad h_n(R) = 2\left(\sum_{\mu=-\infty}^{\infty} - \int_{-\infty}^{\infty} d\mu\right) C_\mu\, e^{-in\vartheta_\mu+iY\sin\vartheta_\mu+iX\cos\vartheta_\mu},$$

where μ is an integer for the sum operation and a continuous variable for the integral; similarly for $X<0$, we replace ϑ_μ by $\pi-\vartheta_\mu$. (Using grating terminology [*4*], we may say that (11) indicates a sum over the discrete plane wave modes of the grating minus the integral over the analogous range of continuous modes.)

Substituting (11) in (5), and letting $Y=0$, $X=\varepsilon\to+0$, we write

$$\mathscr{H}_n = \operatorname*{Lim}_{\varepsilon\to 0} 2\Big(\sum_{\mu=-\infty}^{\infty} - \int_{-\infty}^{\infty} d\mu\Big) e^{i\varepsilon\cos\vartheta_\mu} C_\mu e^{-in\vartheta_\mu} \equiv 2\,\mathsf{S}\, C_\mu e^{-in\vartheta_\mu}. \tag{12}$$

Equivalently, if we first replace X by $-X$ and ϑ_μ by $\pi-\vartheta_\mu$ in (11) and then use (5), we obtain

$$\mathscr{H}_n = 2\,\mathsf{S}\, C_\mu e^{-in(\pi-\vartheta_\mu)}. \tag{13}$$

The arithmetical mean of (12) and (13), *i.e.*,

$$\mathscr{H}_n = \mathsf{S}\, C_\mu [e^{-in\vartheta_\mu} + e^{-in(\pi-\vartheta_\mu)}], \tag{14}$$

satisfies

$$\mathscr{H}_n = (-1)^n \mathscr{H}_{-n}, \tag{15}$$

as obtained directly from (3) by using $H_n=(-1)^n H_{-n}$. Because of (15), we consider explicitly only positive values of n.

In performing the operation $\mathsf{S} f(\mu)$ we split it into parts which are explicitly convergent and parts which require the use of the exponential "convergence factor" $e^{i\varepsilon\cos\vartheta_\mu}$, where $\cos\vartheta_\mu \sim i|\mu|\, 2\pi/D$ as $|\mu|\to\infty$. For the first, we evaluate the resulting integrals and leave the sums unaltered. For the second, we arrange matters so that the operation is carried out essentially on a polynomial, and then use EULER's summation formula to obtain a finite sum. For convenience, we apply EULER's formula in the form given by NÖRLUND [*9*]:

$$\operatorname*{Lim}_{\varepsilon\to 0}\Big\{\int_0^\infty z^{n-1} e^{-z\varepsilon}\, dz - \sum_{s=0}^{\infty} (s+x)^{n-1} e^{-(s+x)\varepsilon}\Big\} = B_n(x)/n, \tag{16}$$

where $B_n(x)$ is the Bernoulli polynomial. Specifically, we shall use the equivalent form

$$\begin{aligned} \mathsf{S}_+ (\mu+\Delta\sin\vartheta_0)^{n-1} &\equiv \operatorname*{Lim}_{\varepsilon\to 0}\Big(\sum_{\mu=0}^{\infty} - \int_{-\Delta\sin\vartheta_0}^{\infty} d\mu\Big)(\mu+\Delta\sin\vartheta_0)^{n-1} e^{i\varepsilon\cos\vartheta_\mu} \\ &= -B_n(\Delta\sin\vartheta_0)/n, \end{aligned} \tag{17}$$

where $\mu=-\Delta\sin\vartheta_0 \equiv -(D/2\pi)\sin\vartheta_0$ corresponds to $\vartheta_\mu=0$. Since $\cos\vartheta_\mu = [1-(\mu+\Delta\sin\vartheta_0)^2/\Delta^2]^{\frac{1}{2}} \sim i(\mu+\Delta\sin\vartheta_0)/\Delta$, the convergence factors in (16) and (17) are equivalent for the purpose at hand.

3. Elementary Function Representations

We rewrite (13) as

$$\mathscr{H}_n = 2\,\mathsf{S}_-\, C_\mu e^{-in(\pi-\vartheta_\mu)} + 2\,\mathsf{S}_+\, C_\mu e^{-in(\pi-\vartheta_\mu)}, \tag{18}$$

where

$$\begin{aligned} \mathsf{S}_- f(\mu) &\equiv \operatorname*{Lim}_{\varepsilon\to 0}\Big(\sum_{\mu=-\infty}^{-1} - \int_{-\infty}^{-\Delta\sin\vartheta_0} d\mu\Big) f(\mu)\, e^{i\varepsilon\cos\vartheta_\mu}, \\ \mathsf{S}_+ f(\mu) &\equiv \operatorname*{Lim}_{\varepsilon\to 0}\Big(\sum_{0}^{\infty} - \int_{-\Delta\sin\vartheta_0}^{\infty} d\mu\Big) f(\mu)\, e^{i\varepsilon\cos\vartheta_\mu}, \qquad \Delta\equiv D/2\pi. \end{aligned} \tag{19}$$

Adding and subtracting $e^{-in\vartheta_\mu}$ in the kernel of S_+ and introducing

$$F_n \equiv 2\,\mathsf{S}_+\,C_\mu\left[e^{-in(\pi-\vartheta_\mu)} - e^{-in\vartheta_\mu}\right], \tag{20}$$

we write

$$\mathscr{H}_n = 2\,\mathsf{S}_-\,C_\mu\,e^{-in(\pi-\vartheta_\mu)} + 2\,\mathsf{S}_+\,C_\mu\,e^{-in\vartheta_\mu} + F_n. \tag{21}$$

In the range of S_- for which $\sin\vartheta_\mu < -1$ we have $\vartheta_\mu = -\frac{1}{2}\pi + i|\eta_\mu|$; similarly in the range of S_+ for which $\sin\vartheta_\mu > 1$ we have $\vartheta_\mu = \frac{1}{2}\pi - i|\eta_\mu|$. Consequently the first two terms on the right-hand side of (20) converge independently of the implicit convergence factor. Thus letting $\varepsilon = 0$ in the first two terms, we evaluate the integrals, *e.g.*,

$$2\int_{-\infty}^{-\Delta\sin\vartheta_0} C_\mu\,e^{in\vartheta_\mu}\,d\mu = \frac{1}{\pi}\int_{-\frac{1}{2}\pi+i\infty}^{0} e^{in\vartheta}\,d\vartheta = \frac{1}{i\,n\,\pi},$$

and leave the sums as they stand to obtain

$$\mathscr{H}_n = e^{-in\pi}\left[2\sum_{-\infty}^{-1} C_\mu\,e^{in\vartheta_\mu} - \frac{1}{i\,n\,\pi}\right] + \left[2\sum_{0}^{\infty} C_\mu\,e^{-in\vartheta_\mu} - \frac{1}{i\,n\,\pi}\right] + F_n. \tag{22}$$

To evaluate F_n, we express the kernel of (20) as a polynomial in $\sin\vartheta_\mu = (\mu + \Delta\sin\vartheta_0)/\Delta$, and then apply EULER'S summation formula as exhibited in (17). Thus for n even, we use

$$\frac{\sin 2n\vartheta}{\cos\vartheta} = \sum_{m=1}^{n} \frac{(-1)^{m-1}\,2^{2m-1}\,(n+m-1)!}{(2m-1)!\,(n-m)!}\,\sin^{2m-1}\vartheta \equiv \sum_{m=1}^{n}[n,m]\sin^{2m-1}\vartheta \tag{23}$$

and the form $\sin\vartheta_\mu = (\mu + \Delta\sin\vartheta_0)/\Delta$ to rewrite (20) as

$$F_{2n} = \frac{4i}{D}\,\mathsf{S}_+\,\frac{\sin 2n\,\vartheta_\mu}{\cos\vartheta_\mu} = \frac{i}{\pi\,\Delta}\sum_{m=1}^{n}\frac{[n,m]}{\Delta^{2m-1}}\,\mathsf{S}_+\,(\mu + \Delta\sin\vartheta_0)^{2m-1}. \tag{24}$$

Applying (17), we obtain

$$F_{2n} = \frac{2i}{\pi\,\Delta}\sum_{m=1}^{n}\frac{[n,m]\,B_{2m}}{\Delta^{2m-1}\,2m}. \tag{25}$$

Similarly for n odd, we use

$$\frac{\cos(2n+1)\,\vartheta}{\cos\vartheta} = \sum_{m=0}^{n}\frac{(-1)^m\,2^{2m}\,(n+m)!}{(2m)!\,(n-m)!}\,\sin^{2m}\vartheta \equiv \sum_{m=0}^{n}\{n,m\}\sin^{2m}\vartheta \tag{26}$$

to write (20) as

$$F_{2n+1} = \frac{-4}{D}\,\mathsf{S}_+\,\frac{\cos(2n+1)\,\vartheta_\mu}{\cos\vartheta_\mu} = \frac{-2}{\pi\,\Delta}\sum_{m=0}^{n}\frac{\{n,m\}}{\Delta^{2m}}\,\mathsf{S}_+\,(\mu + \Delta\sin\vartheta_0)^{2m}. \tag{27}$$

Applying (17), we obtain

$$F_{2n+1} = \frac{2}{\pi\,\Delta}\sum_{m=0}^{n}\frac{\{n,m\}\,B_{2m+1}}{\Delta^{2m}\,(2m+1)}. \tag{28}$$

Thus (22) together with (25) and (28) are the desired elementary function representations of (3) for $n > 0$. (If $n < 0$, we use (15) and the above.) More explicitly

$$\begin{aligned}\mathscr{H}_{2n} &= 2\sum_{s=1}^{\infty} H_{2n}(sD)\cos(sD\sin\vartheta_0) = 2\,\mathsf{S}\,C_\mu \cos 2n\vartheta_\mu \\ &= 2\sum_{0}^{\infty} C_\mu e^{-i2n\vartheta_\mu} + 2\sum_{-\infty}^{-1} C_\mu e^{i2n\vartheta_\mu} + \\ &\quad + \frac{i}{n\pi} + \frac{i}{\pi}\sum_{m=1}^{n} \frac{(-1)^m 2^{2m}(n+m-1)!\,B_{2m}(\varDelta\sin\vartheta_0)}{(2m)!\,(n-m)!\,\varDelta^{2m}},\end{aligned} \tag{29}$$

$$\begin{aligned}\mathscr{H}_{2n+1} &= -2i\sum_{s=1}^{\infty} H_{2n+1}(sD)\sin(sD\sin\vartheta_0) = -2i\,\mathsf{S}\,C_\mu \sin(2n+1)\vartheta_\mu \\ &= 2\sum_{0}^{\infty} C_\mu e^{-i(2n+1)\vartheta_\mu} - 2\sum_{-\infty}^{-1} C_\mu e^{i(2n+1)\vartheta_\mu} + \\ &\quad + \frac{2}{\pi}\sum_{m=0}^{n} \frac{(-1)^m 2^{2m}(n+m)!\,B_{2m+1}(\varDelta\sin\vartheta_0)}{(2m+1)!\,(n-m)!\,\varDelta^{2m+1}},\end{aligned} \tag{30}$$

where $H_n = H_n^{(1)}$ is the Hankel function of the first kind as defined by WATSON [*1*], B_n is the Bernoulli polynomial as defined by NÖRLUND [*9*] and ERDÉLYI *et al.* [*10*], and C_μ, ϑ_μ (in terms of $\varDelta = D/2\pi$) are defined in (8) and discussed in subsequent paragraphs.

4. Special Cases and Applications

Series for $\vartheta_0 = 0$. If $\vartheta_0 = 0$, then the "Hankel series" in (30) vanishes. Equivalently, since for this case $\sin\vartheta_\mu = \mu/\varDelta = -\sin\vartheta_{-\mu}$, and since $B_1(0) = -\frac{1}{2}$ and $B_{2n-1}(0) = 0$ for $n > 0$ [*10*, p. 38], we see that the final form in (30) reduces to $2C_0 + 2B_1/\pi\varDelta = 2/D + 4B_1/D = 0$. Consequently, for this case we obtain

$$\begin{aligned}\mathscr{H}_{2n} &= 2\sum_{s=1}^{\infty} H_{2n}(sD) = \frac{2}{D} + 4\sum_{1}^{\infty} C_\mu e^{-i2n\vartheta_\mu} + \\ &\quad + \frac{i}{n\pi} + \frac{i}{\pi}\sum_{m=1}^{n} \frac{(-1)^m 2^{2m}(n+m-1)!\,B_{2m}(0)}{(2m)!\,(n-m)!\,\varDelta^{2m}},\end{aligned} \tag{31}$$

where $n > 0$. Writing

$$\mathscr{H}_{2n} = 2\sum_{s=1}^{\infty} J_{2n}(sD) + i\,2\sum_{s=1}^{\infty} N_{2n}(sD) \equiv \mathscr{J}_{2n} + i\,\mathscr{N}_{2n}, \tag{32}$$

we separate (31) into the corresponding real "Bessel series" and "Neumann series":

$$\mathscr{J}_{2n} = 2\sum_{s=1}^{\infty} J_{2n}(sD) = \frac{2}{D} + \frac{4}{D}\sum_{\mu=1}^{\mu_+} \frac{\cos 2n\vartheta_\mu}{\cos\vartheta_\mu}, \tag{33}$$

$$\begin{aligned}\mathscr{N}_{2n} &= 2\sum_{s=1}^{\infty} N_{2n}(sD) = \frac{1}{n\pi} + \frac{1}{\pi}\sum_{m=1}^{n} \frac{(-1)^m 2^{2m}(n+m-1)!\,B_{2m}(0)}{(2m)!\,(n-m)!\,\varDelta^{2m}} - \\ &\quad - \frac{4}{D}\sum_{\mu=1}^{\mu_+} \frac{\sin 2n\vartheta_\mu}{\cos\vartheta_\mu} - \frac{4(-1)^n}{D}\sum_{\mu_++1} \frac{e^{-2n\eta_\mu}}{\sinh\eta_\mu},\end{aligned} \tag{34}$$

where $\sin\vartheta_\mu = 2\pi\mu/D = \mu/\varDelta = \cosh\eta_\mu$, $\mu_+ < \varDelta < \mu_+ + 1$. Since $e^{-2n\eta_\mu}/\sinh\eta_\mu = \left[\frac{\mu}{\varDelta} - \sqrt{\left(\frac{\mu}{\varDelta}\right)^2 - 1}\right]^{2n}\left[\left(\frac{\mu}{\varDelta}\right)^2 - 1\right]^{-\frac{1}{2}}$, and since the Bernoulli numbers $B_{2n}(0)$ equal $-(-1)^n B_n$ in IGNATOWSKY'S [*2*] notation, the present series equal those he derived originally [*2*].

More generally, we may write

$$\mathscr{H}_n = \mathscr{J}_n + i\,\mathscr{N}_n, \tag{35}$$

where

$$\mathscr{J}_n \equiv \sum_{s=1}^{\infty} J_n(sD)\,[e^{isD\sin\vartheta_0}(-1)^n + e^{-isD\sin\vartheta_0}] = \mathop{\mathsf{S}}_{p} C_\mu [e^{-in\vartheta_\mu} + e^{-in(\pi-\vartheta_\mu)}], \tag{36}$$

$$\mathscr{N}_n \equiv \sum_{s=1}^{\infty} N_n(sD)\,[e^{isD\sin\vartheta_0}(-1)^n + e^{-isD\sin\vartheta_0}] = \frac{1}{i} \mathop{\mathsf{S}}_{e} C_\mu [e^{-in\vartheta_\mu} + e^{-in(\pi-\vartheta_\mu)}]. \tag{37}$$

Here the subscripts p and e designate that the operation is restricted to the propagating range ($|\sin\vartheta_\mu| < 1$) and evanescent range respectively. Thus, for all n, these "Bessel series" and "Neumann series" are associated with the propagating and evanescent ranges respectively.

Bessel series. Elementary function representations of the Bessel series are obtained by inspection of (29) and (30) or by direct application of

$$\mathop{\mathsf{S}}_{p} f(\mu) = \left(\sum_{-\mu_-}^{\mu_+} - \int_{-m_-}^{m_+} d\mu\right) f(\mu) = \left(\sum_{-\mu_-}^{\mu_+} - \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{d\vartheta_\mu}{2\pi C_\mu}\right) f(\mu).$$

Thus

$$\begin{gathered} \mathscr{J}_{2n} = 2\sum_{s=1}^{\infty} J_{2n}(sD)\cos(sD\sin\vartheta_0) = \frac{2}{D}\sum_{-\mu_-}^{\mu_+} \frac{\cos 2n\vartheta_\mu}{\cos\vartheta_\mu} - \delta_{n0}, \\ \delta_{00} = 1, \quad \delta_{n0} = 0 \quad \text{if} \quad n > 0, \end{gathered} \tag{38}$$

$$\mathscr{J}_{2n+1} = -2i\sum_{s=1}^{\infty} J_{2n+1}(sD)\sin(sD\sin\vartheta_0) = \frac{-2i}{D}\sum_{-\mu_-}^{\mu_+} \frac{\sin(2n+1)\vartheta_\mu}{\cos\vartheta_\mu} \tag{39}$$

(where we included the case $n=0$ for completeness). These finite-sum representations for $\mathscr{J}_n$ (whose derivation does not involve the limiting process indicated in (5) and (21)) may of course also be obtained by various more or less elementary procedures.

Neumann series. Perhaps the simplest representations for $\mathscr{N}_n$ are those left in the form $\mathscr{H}_n - \mathscr{J}_n$ with the $\mathscr{H}$'s as in (29) and (30) and the $\mathscr{J}$'s as in (38) and (39). (In any event, it is somewhat simpler to obtain $\mathscr{N}_n$ by first deriving $\mathscr{H}_n$ as we have done than to apply S_e as in (37).) However, it is convenient for computations to have these components explicitly. Thus

$$\begin{aligned} \mathscr{N}_{2n} &= 2\sum_{s=1}^{\infty} N_{2n}(sD)\cos(sD\sin\vartheta_0) \\ &= \frac{1}{n\pi} + \frac{1}{\pi}\sum_{m=1}^{n} \frac{(-1)^m 2^{2m}(n+m-1)!\,B_{2m}(\varDelta\sin\vartheta_0)}{(2m)!\,(n-m)!\,\varDelta^{2m}} - \\ &\quad - \frac{2}{D}\left(\sum_{0}^{\mu_+} - \sum_{-1}^{-\mu_-}\right)\frac{\sin 2n\vartheta_\mu}{\cos\vartheta_\mu} - \frac{2(-1)^n}{D}\left[\sum_{\mu_+ +1}^{\infty} \frac{e^{-2n\eta_\mu^+}}{\sinh\eta_\mu^+} + \sum_{\mu_- +1}^{\infty} \frac{e^{-2n\eta_\mu^-}}{\sinh\eta_\mu^-}\right] \end{aligned} \tag{40}$$

where

$$\cosh\eta_\mu^\pm = \pm\sin\vartheta_0 + \mu/\varDelta = \pm\sin\vartheta_0 + 2\pi\mu/D.$$

Similarly

$$\mathscr{N}_{2n+1} = -2i \sum_{s=1}^{\infty} N_{2n+1}(sD)\sin(sD\sin\vartheta_0)$$

$$(41)\qquad = \frac{-2i}{\pi}\sum_{m=0}^{n} \frac{(-1)^m\, 2^{2m}\,(n+m)!\, B_{2m}(\varDelta\sin\vartheta_0)}{(2m+1)!\,(n-m)!\,\varDelta^{2m+1}} -$$

$$-\frac{2i}{D}\left(\sum_{0}^{\mu_+} - \sum_{-1}^{-\mu_-}\right)\frac{\cos(2n+1)\vartheta_\mu}{\cos\vartheta_\mu} + \frac{2i(-1)^n}{D}\left[\sum_{\mu_++1}^{\infty}\frac{e^{-(2n+1)\eta_\mu^+}}{\sinh\eta_\mu^+} - \sum_{\mu_-+1}^{\infty}\frac{e^{-(2n+1)\eta_\mu^-}}{\sinh\eta_\mu^-}\right].$$

It is sometimes convenient to suppress the $\mu=0$ term of (40) or (41) by replacing B_m by $B_m + \frac{m}{2}(\varDelta\sin\vartheta_0)^{m-1}$ in the sum over m.

For completeness, we include

$$(42)\qquad \begin{aligned} \mathscr{N}_0 = &-\frac{2}{\pi}\log\frac{\gamma D}{4\pi} + \frac{1}{\pi}\left(\sum_{1}^{\mu_+} + \sum_{1}^{\mu_-}\right)\frac{1}{\mu} - \\ &- 2\sum_{\mu_++1}^{\infty}\left[\frac{1}{D\sinh\eta_\mu^+} - \frac{1}{2\pi\mu}\right] - 2\sum_{\mu_-+1}^{\infty}\left[\frac{1}{D\sinh\eta_\mu^-} - \frac{1}{2\pi\mu}\right] \end{aligned}$$

as in [6, p. 60] or as obtained by the present procedure and EULER's result: $\left(\sum_{1}^{\infty} - \int_{1}^{\infty} d\mu\right)\frac{1}{\mu} \equiv \log\gamma$, $\gamma = 1.781\ldots$. (For this case we set $\varepsilon = 0$ in (37), carry out the operations for a finite upper limit M, and then let $M\to\infty$.)

Related series. The above results may be applied directly to obtain elementary function forms for related Schlömilch series. Thus using the recursive relations for the cylindrical functions, we have

$$(43)\qquad \frac{4(2n+1)}{D}\sum_{s=1}^{\infty}\frac{H_{2n+1}(sD)}{s}\cos(sD\sin\vartheta_0) = \mathscr{H}_{2n} + \mathscr{H}_{2n+2},$$

$$(44)\qquad 2\sum_{s=1}^{\infty} H'_{2n+1}(sD)\cos(sD\sin\vartheta_0) = \mathscr{H}_{2n} - \mathscr{H}_{2n+2},\quad H'_n(x) = \frac{d}{dx}H_n(x),$$

etc. Additional results follow from the trigonometric identities and by differentiation with respect to $\sin\vartheta_0$.

Expansions for $D(1\pm\sin\vartheta_0)/2\pi<1$. If there is only one propagating mode, *i.e.*, $D(1\pm\sin\vartheta_0)/2\pi<1$ (which is of physical interest in the scattering of wavelengths larger than the grating spacing, and in related waveguide problems), then the Bessel function series reduce to

$$(45)\qquad \mathscr{J}_{2n} = \frac{2\cos 2n\vartheta_0}{D\cos\vartheta_0} - \delta_{n0},\qquad \mathscr{J}_{2n+1} = -\frac{2i\sin(2n+1)\vartheta_0}{D\cos\vartheta_0}.$$

In this range, the leading Neumann function series up to terms of the order D^2 are given by

$$(46)\qquad \begin{aligned} \mathscr{N}_0 &\approx -\frac{2}{\pi}\log\frac{\gamma D}{4\pi} - \frac{D^2}{2\pi^3}\left(\frac{1}{2} + \sin^2\vartheta_0\right)1.202, \\ \mathscr{N}_1 &\approx \frac{i\,2\sin\vartheta_0}{\pi} - \frac{i\,2D^2\sin\vartheta_0}{4\pi^3}1.202, \\ \mathscr{N}_2 &\approx -\frac{4\pi}{3D^2} + \frac{1}{\pi}(1-2\sin^2\vartheta_0) + \frac{D^2}{8\pi^3}1.202, \\ \mathscr{N}_3 &\approx \frac{i\,16\pi\sin\vartheta_0}{3D^2} - \frac{i}{\pi}2\sin\vartheta_0\left(1-\frac{4}{3}\sin^2\vartheta_0\right), \\ \mathscr{N}_4 &\approx \frac{-2^5\pi^3}{15D^4} - \frac{16\pi}{D^2}\left(\frac{1}{6} - \sin\vartheta_0\right) + \frac{1}{2\pi}(1-8\sin^2\vartheta_0+8\sin^4\vartheta_0). \end{aligned}$$

Here we have used the fact that $\sum_{1}^{\infty} \mu^{-3} \approx 1.202$. The next terms in the above are of order D^4. (Note that $\mathcal{J}_0$ and $\mathcal{J}_1$ are proportional to D^{-1} and consequently tend to infinity more rapidly than $\mathcal{N}_0$ and $\mathcal{N}_1$ as $D \to 0$.)

The leading terms of $\mathcal{N}$ for large n are given by

$$\mathcal{N}_{2n} \approx \frac{(-1)^n\, 2^{2n-1}\, B_{2n}(0)}{\pi\, n\, (D/2\pi)^{2n}}, \tag{47}$$

$$\mathcal{N}_{2n+1} \approx \frac{(-1)^n\, 2^{2n+1} \sin\vartheta_0\, B_{2n}(0)}{i\,\pi\,(D/2\pi)^{2n}}. \tag{48}$$

IGNATOWSKY [*2*] showed for the case $\vartheta_0 = 0$ that the leading term of $\mathcal{N}_{2n}$ for $D \to 0$ could also be obtained from the original Neumann function series by heuristically using the leading term of $N_n(t)$ for $n \gg t$, *i.e.*, by using $N_n(t) \approx -\frac{(n-1)!\,2^n}{\pi\, t^n}$. Similarly for arbitrary ϑ_0, we find for $D \to 0$,

$$\mathcal{N}_{2n} = 2\sum_{s=1}^{\infty} N_{2n}(sD)\cos(sD\sin\vartheta_0) \approx -\frac{2(2n-1)!\,2^{2n}}{\pi\, D^{2n}} \sum_{s=1}^{\infty} \frac{1}{s^{2n}} = (47),$$

$$\mathcal{N}_{2n+1} = -2i\sum_{s=1}^{\infty} N_{2n+1}(sD)\sin(sD\sin\vartheta_0) \approx \frac{2i\,D\sin\vartheta_0\,(2n)!\,2^{2n+1}}{\pi\, D^{2n+1}} \sum_{s=1}^{\infty} \frac{1}{s^{2n}} = (48),$$

where the final forms represent the right-hand sides of (47) and (48).

Near grazing mode. Another range of physical interest corresponds to a "discrete mode near grazing", *i.e.*, to $\vartheta_{\mu+}$ (the "last" propagating mode) or $\vartheta_{\mu_+ +1}$ (the "first" evanescent mode) approaching $\frac{1}{2}\pi$. If a propagating mode is near grazing, then (22) yields

$$\mathcal{H}_n = G_n + \frac{2i^{-n}}{D\,\varepsilon}, \qquad \vartheta_{\mu+} = \tfrac{1}{2}\pi - \varepsilon, \qquad 0 < \varepsilon \ll 1, \tag{49}$$

where G_n is finite in the limit. On the other hand, if an evanescent mode is near grazing, then

$$\mathcal{H}_n = G_n + \frac{2i^{-n}}{i\,D\,\varepsilon}, \qquad \vartheta_{\mu_+ +1} = \tfrac{1}{2}\pi - i\,\varepsilon. \tag{50}$$

Similarly if $\vartheta_{\mu_-} \to -\frac{1}{2}\pi$, we obtain i^n instead of i^{-n}. If both $\vartheta_{\mu+}$ and $\vartheta_{\mu-}$ are near grazing, then we replace i^{-n} in the above by $2\cos(n\pi/2)$; thus for n even, the isolated terms involve $2(-1)^n$ instead of i^{-n}, and for n odd, they vanish. The factor of i which distinguishes the second case from the first is significant in connection with the "Wood anomalies", or "grating resonances" [*4*]; it is related to maxima of the scattered field occurring for an evanescent mode near grazing.

Asymptotic form for $D \gg n$. If $D \gg n$, then on introducing the usual asymptotic form of $\mathcal{H}_n$ directly in (3) we obtain

$$\begin{aligned} &\mathcal{H}_n \sim i^{-n}[\mathcal{H}_+(-1)^n + \mathcal{H}_-], \\ &\mathcal{H}_\pm = \sqrt{\frac{2}{i\,\pi D}} \sum_{s=1}^{\infty} \frac{e^{isD(1\pm\sin\vartheta_0)}}{\sqrt{s}}, \end{aligned} \tag{51}$$

where $1/\sqrt{i}=e^{-i\frac{1}{4}\pi}$; the next terms of $\mathscr{H}_n$ involve $(s\,D)^{-\frac{3}{2}}$. Using Euler's summation formula, we write

$$\begin{aligned}\mathscr{H}_{\pm}&\sim\sqrt{\frac{2}{D\,\delta_{\pm}}}\left[1-\sqrt{\frac{2}{i}}\,F(\delta_{\pm})\right]+\sqrt{\frac{2}{i\,\pi\,D}}\,e^{i\,\delta_{\pm}}\left[\frac{1}{2}+\frac{1}{12}\left(\frac{1}{2}-i\,\delta_{\pm}\right)+\cdots\right],\\ \delta_{\pm}&\equiv D\,(1\pm\sin\vartheta_0)-2\nu_{\mp}\,\pi=2\pi\,(m_{\mp}-\nu_{\mp}),\end{aligned}\tag{52}$$

where ν_+ and ν_- are integers, and where F is the Fresnel integral[3]. (See [5] for detailed treatment of (51) for a finite upper limit.) Thus if $\delta_{\pm}$ is not small, then $\mathscr{H}_{\pm}$ is of order $D^{-\frac{1}{2}}$. In particular, if $\delta_{\pm}$ is large, then using $F(x)\sim\sqrt{i/2}+e^{i\,x}/\sqrt{2\pi\,x}$ shows that the first term of (52) which is then of order $D^{-\frac{3}{2}}$ is to be dropped for consistency.

On the other hand if $\delta_{\pm}\to 0$ (*i.e.*, if a discrete mode is near grazing), then using $F(x)\approx\sqrt{2\,x/\pi}$, we obtain

$$\mathscr{H}_{\pm}\approx\sqrt{\frac{2}{D\,\delta_{\pm}}}+G_{\pm}=\frac{1}{D}\sqrt{\frac{2}{1\pm\sin\vartheta_{\nu\mp}}}+G_{\pm}\approx\frac{2}{D\cos\vartheta_{\nu\pm}}+G_{\pm},\tag{53}$$

where $G_{\pm}$ is finite in the limit. Thus identifying ν_+ with μ_+ and $\mu_+ + 1$ of (49) and (50), *etc.*, we see that the correct limiting forms for near grazing modes follow directly from the asymptotic representation (51).

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[3] The present function $F(u)$ equals $C(u)+i\,S(u)$ in terms of the functions tabulated in [11].

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Motions in the Euclidean Plane under the Action of some Harmonic Gradient Fields

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Introduction

Probably the simplest problem in classical mechanics is that of the motion of a material point or particle which moves under the influence of a constant force field in Euclidean space. Written in vector form, the equations for such a motion in n-dimensional Euclidean space are

$$\ddot{x} = \alpha \tag{1}$$

where $x = (x^1, x^2, \ldots, x^n)$, $\alpha = (\alpha^1, \alpha^2, \ldots, \alpha^n)$, and α represents a constant vector[1].

The force vector α of the system (1) represents a contravariant vector field in n-dimensional Euclidean space and as such can be considered as an infinitesimal transformation on the space. In particular, a constant vector is an infinitesimal translation. The motions considered here are those under the action of force fields that are generalizations of the infinitesimal translations.

The generalizations are made in two ways. One way depends on some group-theoretic properties [*3*] [*4*] and provides a generalization in non-Euclidean spaces. The other way in which the generalization is made depends on potential-theoretic considerations. The motions discussed here are those under the action of force fields which are generalizations of the infinitesimal translations made in this latter way.

An infinitesimal translation in Euclidean space is the gradient of the function $u = \sum_{i=1}^{n} \alpha^i x^i$, which is harmonic, vanishes on a hyperplane and is positive on one side of the hyperplane. Now let D be a convex domain in the Euclidean plane. One would like to find a function V harmonic in D, which is positive in D and vanishes on ∂D (the boundary of D). Clearly one cannot expect to have V harmonic everywhere in D unless, of course, the requirement that V vanish everywhere on ∂D is dropped. There are two possibilities for V that immediately suggest themselves. If $V = G(P, Q)$, where G is the Green's function of D, then V satisfies the requirements in that it is harmonic, vanishes on ∂D and is positive in D. Also, if $w = f(z) = u + iv$ is the analytic function which maps D onto the upper-half plane, $v > 0$, with ∂D going onto the real axis, then the function v has the property that it is harmonic in D, vanishes on ∂D, with the exception of the point which goes into infinity, and is positive in D. Thus one obtains a generalization of the infinitesimal translation by considering the gradient fields of v or V as infinitesimal transformations. In particular the motions to be considered are those governed by the equations

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[1] Dots will be used to represent differentiation with respect to time throughout the remainder of this paper. Also it is assumed, without loss of generality, that the particle has unit mass.

$$\begin{aligned} \ddot{x} &= -v_x, \\ \ddot{y} &= -v_y. \end{aligned} \tag{2}$$

In 1954 FIREY [2] introduced the concept of ballistically closed sets associated with a system of equations

$$\ddot{x} = A x, \tag{3}$$

where $x(t)$ is a vector function in a real Euclidean vector space and A is a bounded linear operator. In particular, a closed set F with interior point is said to be ballistically closed with respect to (3) if (i) for two points $P_1, P_2, \in F$ and two distinct times t_1 and t_2 there exists a unique solution to (3) such that $x(t_1) = P_1$, $x(t_2) = P_2$; (ii) the path segment between P_1 and P_2 lies entirely in F.

The concept of ballistic closure can easily be introduced for a system of equations of the form $\ddot{x} = f(x, \dot{x})$. Indeed, if F is defined by $\sum_{i=1}^{n} \alpha^i x^i \leq 0$, then the system (1) is ballistically closed. It is easily shown that in general (2) is not ballistically closed with respect to D. This is done by considering a problem where criteria established by FIREY can be applied.

Equations of Motion and Existence of Solutions

Let D be a convex domain in the Euclidean plane whose boundary contains the point at infinity. It is easily shown that for such a domain there exists at least one direction such that every line in that direction intersects $\bar{D}$ in at least a half-line or not at all. Thus without loss of generality it is assumed that the intersection of every vertical line with $\bar{D}$ is either a half-line (possibly the entire line) or is empty. Further it is assumed that the half-line extends to infinity in the direction of the positive y-axis. Let $f(z) = u(x, y) + i v(x, y)$ be an analytic function which maps D conformally onto the upper half-plane $v > 0$ with the point at infinity going into infinity and the remainder of ∂D going into $v = 0$. Let $\vec{F} = -\operatorname{grad} v$; then $\vec{F}$ is a generalized infinitesimal translation and the equations of motion are given by (2). By a solution of (2) are meant two functions $x(t)$ and $y(t)$, denoted by $z(t)$, which satisfy (2) and are in D. Also assume that $t > 0$.

Equations (2) yield the energy integral

$$\frac{\dot{x}^2 + \dot{y}^2}{2} = -v + c. \tag{4}$$

Since $v(x, y) > 0$ in D, then $0 \leqq v \leqq c$. Thus one concludes from (4) that

$$\dot{x} \leqq \sqrt{c}, \quad \dot{y} \leqq \sqrt{c} \tag{5}$$

for $z(t)$ in D.

Since the function $v(x, y)$ is harmonic in D, v_x and v_y satisfy a Lipschitz condition in every compact subdomain of D. Consequently there exists a unique solution for some interval of time for any given initial conditions.

Proposition 1. *The solution $z(t)$ either goes to ∂D or the solution exists for all time.*

Proof. Suppose that $z(t)$ does not go to ∂D in finite time. Let t_0 be the supremum of all $t \geqq 0$ for which the solution exists and suppose that t_0 is finite. It is evident from (5) that $z(t)$ will approach a finite limiting position as $t \to t_0$. From the continuity of v_x and v_y in D and equations (2), $x(t)$ and $y(t)$ will approach a finite limit as $t \to t_0$. Thus $x(t)$ and $y(t)$ will approach limiting values at t_0. Consequently, the solution $z(t)$ will exist for some $t > t_0$. This contradicts the assumption that t_0 was the supremum of all such t. Hence t_0 must be infinite.

Before attempting to discuss the motions governed by equations (2), it is necessary to establish some preliminary results concerning the function $v(x, y)$ and its behavior in D.

Behavior of $v(x, y)$ in D

If again $f(z) = u(x, y) + iv(x, y)$ is an analytic univalent function that maps D conformally onto the upper half-plane $v > 0$ and takes the point at infinity into the point at infinity, then the Green's function of D is

$$G(z, z_0) = -\log\left|\frac{f(z)-f(z_0)}{f(z)-\bar{f}(z_0)}\right|$$

or

$$G(z, z_0) = -\frac{1}{2}\log\left[1-\frac{4v\,v_0}{(u-u_0)^2+(v+v_0)^2}\right].$$

Thus it is seen that there is a continuous real-valued function

$$\varrho(z, z_0) = \frac{(u-u_0)^2+(v+v_0)^2}{v_0}$$

such that for each $z \in D$

$$\lim_{z_0\to\infty} \varrho(z, z_0)\, G(z, z_0) = 2v(x, y).$$

Since

$$\varrho(z, z_0) = \left[\frac{(u-u_0)^2+(v+v_0)^2}{v_0^2+u_0^2}\right]\left[\frac{u_0^2+v_0^2}{v_0}\right]$$

set $\varrho(z, z_0) = \sigma(z, z_0)\,\tau(z_0)$ where $\tau(z_0) = \dfrac{u_0^2+v_0^2}{v_0}$; then

$$\lim_{z_0\to\infty} \tau(z_0)\, G(z, z_0) = 2v(x, y),$$

and the convergence is uniform in any compact subdomain of D. Thus for each compact subdomain of D we can find a sequence $\{z_{0n}\}$ such that $z_{0_n} \to \infty$ and

$$\lim_{n\to\infty} \tau(z_{0n})\, G(z, z_{0n}) \equiv 2v(x, y).$$

Since $\tau(z_{0_n}) > 0$, then for each n the level curves of $\tau(z_{0_n})\, G(z, z_{0_n})$ are strictly convex [5], so that in the limit the level curves of the function $2v(x, y)$ are convex though not necessarily strictly convex. Since this is true for every compact subdomain of D, the level curves of $v(x, y)$ must be convex throughout D.

There are two domains which are excluded from the present discussion. If D is a half plane, then the equations of motion are given by (1), and the resulting motions are well known. The vertical infinite strip is also excluded and treated separately later. With these two domains excluded, it can be assumed, because of the convexity of D, that D is contained in an angular domain bounded below by two intersecting lines, one with positive slope and one with negative slope.

The domains D have been taken in such a way that the intersection of each vertical line with D is either a half-line or is empty. Let L be the collection of all vertical lines intersecting D in a half-line. If $l_1 \in L$, then the sequence z_{0_n} can be taken to lie on l_1. Since the level curves of $\tau(z_{0_n})\, G(z, z_0)$ are strictly convex, they will intersect l_1 in precisely two points with the point z_{0_n} between them. Thus as $z_{0_n} \to \infty$ along l_1, the level curves of $v(x, y)$ must either intersect l_1 in one point or coincide with l_1. Since this is true for all $l \in L$, the level curves of $v(x, y)$ must intersect every l in one point. Thus since the curve $v(x, y) = k > 0$ is convex, its slope with respect to the x-axis must be finite everywhere in D. The slope is given by

$$\frac{dy}{dx} = -\frac{v_x}{v_y}. \tag{6}$$

Since v_x and v_y never vanish simultaneously in D, equation (6) shows that v_y can never vanish in D. Since $v > 0$ in D and $v = 0$ on ∂D, then $v_y > 0$ in D.

Let $F(w)$ be the inverse of $f(z)$; then $F(w)$ is analytic and univalent, and $F'(w) \neq 0$ for $v > 0$. Thus the function

$$g(w) = \frac{F''(w)}{F'(w)}$$

is regular analytic, and $\operatorname{Im}\{g(w)\}$ is harmonic for $v > 0$. The condition that the curve $v = k$ be convex becomes $\operatorname{Im}\{g(w)\} \geqq 0$. One has the strict convexity with strict inequality. If the maximum principle is applied, then either $\operatorname{Im}\{g(w)\} > 0$ or is identically zero. In the case when

$$\operatorname{Im}\left\{\frac{F''(w)}{F'(w)}\right\} = 0,$$

then

$$\frac{F''(w)}{F'(w)} = a \tag{7}$$

where a is a real constant. If $a = 0$, then $F''(w) = 0$, or $F(w) = \alpha w + \beta$. This implies that D is a half-plane, which has been excluded. If $a \neq 0$, then (7) yields

$$F(w) = \frac{b}{a}\, e^{a w} + c$$

which, contrary to hypothesis, is not univalent for $v > 0$. Thus $\operatorname{Im}\{g(w)\} > 0$, and the level curves $v(x, y) = k$ are convex. A simple calculation gives

$$v_x v_{xx}^2 + 2 v_x v_y v_{xy} + v_y^2 v_{yy} > 0 \tag{8}$$

in D.

Since D is contained in a domain bounded below by two intersecting lines, one with positive slope and one with negative slope, the level curve $v(x, y) = k$ must have positive slope somewhere and negative slope somewhere in D. Thus along any level curve, equation (6) implies that v_x will be both positive and negative. From the strict convexity of the curves, there will be a unique point on each level curve such that $v_x(x, y) = 0$. From (8) it is clear that when $v_x = 0$, $v_{yy} > 0$ in D. Since $v(x, y)$ is harmonic, it follows that $v_{xx} < 0$ in D. Thus $v_x(x, y) = 0$

defines a curve with

$$\frac{dx}{dy} = -\frac{v_{xy}}{v_{xx}}.$$

Since v_{xx} never vanishes in D, $v_x(x, y) = 0$ defines x as a single-valued analytic function of y.

It is still necessary to determine the behavior of the curve $v_x(x, y) = 0$ near ∂D. By the convexity of D and the way in which we have situated the domain D in the xy-plane, there will be a unique horizontal line of support; call it λ_1 [*1*]. Again by the convexity of D, λ_1 will intersect ∂D in one point or in a single line segment. If λ_2 is any horizontal line above λ_1, then λ_2 will intersect ∂D in two points, and there will be points of D lying between λ_1 and λ_2.

If λ_1 intersects ∂D in one point, then let P be any point of D between λ_1 and λ_2. There is a level curve of $v(x, y)$ passing through P and intersecting λ_2 in two distinct points. This level curve is strictly convex and consequently has a unique minimum between λ_1 and λ_2. Thus $v_x(x, y) = 0$ somewhere in the region bounded by λ_2 and ∂D. Since this is true for all such horizontal lines λ_2, and since y decreases monotonically along the curve $v_x(x, y) = 0$, this curve must approach the point of intersection of λ_1 and ∂D.

If λ_1 intersects ∂D in a line segment, the above procedure fails since the domain between λ_2 and ∂D approaches a line segment instead of a point as λ_2 approaches λ_1. This procedure does show that the curve $v_x(x, y) = 0$ must approach λ_1. However, it can be shown that $v_x(x, y) = 0$ must again approach a definite point.

Since y decreases monotonically along the curve $v_x(x, y) = 0$, the curve will either approach a definite point or oscillate and come arbitrarily close to every point of some subinterval of the intersection of λ_1 and ∂D. Suppose $v_x(x, y) = 0$ oscillates and comes arbitrarily close to such a subinterval which is denoted by $\hat{\lambda}_1$.

The function $v_x(x, y)$ is harmonic in D and vanishes on the horizontal line segment λ_1, and can therefore be extended to be harmonic below λ_1. Call this extension $\omega(x, y)$. Let P be an interior point of the segment $\hat{\lambda}_1$. Then there exists an analytic function $g(z)$ such that $\omega = \operatorname{Re}\{g(z)\}$ and such that $g(P) = 0$. It is known that there will be k curves passing through P for which $\omega(x, y) = 0$, k being the order of the zero of $g(z)$ at P, and such that the angle between these curves will be π/k. It is also known that in a neighborhood of P that $\omega(x, y) = 0$ only along these curves. If $k = 1$, then the line segment $\hat{\lambda}_1$ is the only curve in the neighborhood of P for which $\omega(x, y) = 0$. This contradicts the fact that the curve $v_x(x, y) = 0$ comes arbitrarily close to P. Thus $k \geqq 2$, and $g'(P) = 0$. Since this must hold for every P in $\hat{\lambda}_1$, then $g'(z) = 0$ and $g(z)$ must be constant. This contradicts the fact that $\omega(x, y)$ is not constant. Consequently, $v_x(x, y) = 0$ cannot approach a line segment and must approach a definite point on the intersection of λ_1 and ∂D. Therefore the curve $v_x(x, y) = 0$ divides D into two regions D_1 and D_2. In D_1 $v_x > 0$, and $v_x < 0$ in D_2.

Behavior of the Trajectories

It is clear from equations (2) that $\ddot{y} < 0$ in D since $v_y > 0$ in D. Thus $\dot{y}(t)$ is a strictly decreasing function of t. If $\dot{y}(t)$ ever becomes negative, then $\dot{y}(t) < -\eta < 0$ (η a fixed constant) for all subsequent time and $y(t) \leqq -\eta t + \delta$ (δ a constant). Since the right-hand side of this inequality tends to $-\infty$ as $t \to \infty$, and since D is bounded from below in the y-direction, it is easy to see that the solution exists only for a finite interval of time $[0, t_0]$. From the earlier discussion one can further conclude that $z(t)$ tends to a definite boundary point as $t \to t_0$. If $\dot{y}(t)$ vanishes for some value of t, then since $\ddot{y}(t) < 0$, $y(t)$ must become negative in D, and $z(t)$ will behave as above.

Now suppose that $\dot{y}(t)$ is positive throughout the motion of the particle. Assume that the trajectory never approaches the boundary of D. Let $z(0) = z_0$ be the initial position of $z(t)$. Then construct a region R bounded by the curve $v(x, y) = c$, ∂D and the two vertical lines $x = \pm x_1$ $(x_1 > 0)$ such that the portion of the curve $v_x(x, y) = 0$ between ∂D and $v(x, y) = c$ lies completely in R and z_0 also lies in R. Since it has been assumed that the boundary is not approached, then there exists a sufficiently narrow strip along ∂D in R which will never be entered. Delete this strip from R and one obtains a region $\widehat{R}$. In $\widehat{R}$, $v_y > \delta > 0$ (δ a fixed constant), and therefore $\ddot{y}(t) < -\delta$ for $z(t)$ in $\widehat{R}$. Thus $\dot{y}(t) < -\delta t + \eta$ (η a constant). This inequality together with the assumption that $\dot{y}(t)$ is always positive is only possible if $z(t)$ leaves $\widehat{R}$. However, $z(t)$ can leave $\widehat{R}$ only by crossing $x = x_1$ or $x = -x_1$. Suppose $z(t)$ crosses $x = x_1$. When $z(t)$ crosses $x = x_1$, $\dot{x}(t) \geqq 0$ and $\ddot{x}(t) > 0$. Therefore $\dot{x}(t)$ must become positive, and $z(t)$ will never re-enter $\widehat{R}$ since to the right of $x = x_1$, $\ddot{x}(t) > 0$, and $x(t)$ will be strictly increasing for all subsequent time.

Now consider the trajectory after $z(t)$ has left R. The slope of the trajectory is given by

$$\frac{dx}{dy} = \frac{\dot{y}}{\dot{x}} .$$

Since $\dot{y}(t)$ is positive and decreasing while $\dot{x}(t)$ is positive and increasing, then dy/dx is positive and decreasing. Therefore the trajectory is concave with respect to the x-axis. Since $\dot{x}(t)$ remains above a fixed positive constant, and since the trajectory is concave while the boundary is convex, $z(t)$ reaches the boundary in finite time. This contradicts the assumption that $z(t)$ did not approach the boundary.

Since $z(t)$ must approach ∂D, it must approach either ∂D_1 or ∂D_2. Assume that $z(t)$ approaches ∂D_2; then $\dot{x}(t)$ must become positive. Thus a discussion similar to the above shows that $z(t)$ reaches the boundary in finite time. Summarizing, one has

Proposition 2. *Let D be a convex domain which is not an infinite strip, and let the point at infinity be a boundary point of D. If $v(x, y)$ is a positive harmonic function in D with $v(x, y) = 0$ on ∂D except at infinity, where it is unbounded, then each solution of equations* (2) *will converge to a definite boundary point in finite time.*

It should be noted that there is essentially only one such function $v(x, y)$ for any given domain D.

The Infinite Strip

The special case of the infinite strip must still be treated. There is no loss of generality if D is taken to be $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$. Thus $v(x, y) = e^y \cos x$ and equations (2) can be written

$$\ddot{x} = e^{y} \sin x, \tag{9}$$

$$\ddot{y} = -e^{y} \cos x. \tag{10}$$

The curve $v_x(x, y) = 0$ is given by $x = 0$. Thus D_1 becomes $-\frac{1}{2}\pi < x < 0$ and D_2, $0 < x < \frac{1}{2}\pi$. One solution to (9) is given by $x(t) = 0$. In this case (10) gives

$$\ddot{y} = -e^{y},$$

which has as its solution

$$e^{y} = c \operatorname{sech}^2 \sqrt{c}\, \frac{(t-\alpha)}{\sqrt{2}}, \qquad c > 0.$$

Thus y tends to minus infinity as $t \to \infty$.

Now let $z(t)$ be a general solution of equations (9) and (10). If $\dot{x}(t)$ ever becomes negative in D_1, then since $\ddot{x}(t) < 0$ in D_1 the solution remains in D_1. Also since $\dot{x}(t) < -\delta$ (δ a positive constant) $z(t)$ will meet the boundary in finite time. Similarly, the solution $z(t)$ will meet ∂D if $\dot{x}(t)$ becomes positive in D_2.

Assume now that $\dot{x}(t)$ is positive if $z(t)$ is in D_1 and is negative if $z(t)$ is in D_2. Under this assumption a solution can never reach the line $x = 0$ in finite time. If this should happen at $t = t_2$, we should have $x(t_2) = 0$ and $\dot{x}(t_2) = 0$. But equation (9) considered as an equation of $x(t)$ alone is of second order and is satisfied by $x(t) \equiv 0$. Since $x(t_2) = 0$ and $\dot{x}(t_2) = 0$, the uniqueness theorem for differential equations guarantees that $x(t) = 0$. The solution therefore remains completely in D_1 or D_2. In this case, considerations similar to previous ones lead to the result that $z(t)$ is defined for all $t > 0$, and $x(t) \to 0$ while $y(t) \to -\infty$ as $t \to \infty$.

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On the Solution of $\Delta u - q(x)u = f(x,y)$ *over a Rectangular Domain*

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1. Introduction*

Let an open rectangle G with boundary Γ be defined in $0<x<a$, $0<y<b$. The boundary-value problem under consideration is to find a solution $u(x, y)$ such that $u=0$ on Γ is twice continuously differentiable in G, satisfies the equation

$$\Delta u - q(x)\,u = f(x, y) \tag{1}$$

everywhere in G, and is such that the improper integral [*1*]

$$\iint_G (u_x^2 + u_y^2 + u^2)\,dx\,dy$$

exists. The functions $f(x, y)$, $q(x)$ and their first partial derivatives are assumed to be Hölder-continuous in $G+\Gamma$, and $q(x)\geqq 0$ in $G+\Gamma$. A particular case of this problem is to find the solution of REYNOLDS' equation [*2*] for partial journal bearings, which can be reduced to (1) by a simple transformation of the function under consideration.

It is known [*1*] that a unique solution to this boundary-value problem exists and is the unique solution of the following variational problem: To find a solution function $u(x, y)$, vanishing on Γ, which minimizes the variational expression

$$J[\psi] \equiv \iint_G \{\psi_x^2 + \psi_y^2 + q(x)\,\psi^2 + 2f(x, y)\,\psi\}\,dx\,dy. \tag{2}$$

The admissible class of functions $\psi(x, y)$ for comparison consists in those functions that are continuous in G, that have piecewise continuous derivatives in G, and that vanish on Γ. Furthermore, the improper integral

$$\iint_G (\psi_x^2 + \psi_y^2 + \psi^2)\,dx\,dy$$

exists. Conversely, the unique solution to the variational problem is the solution of our boundary-value problem.

Based on these known results, together with those of HOPF [*3*], the purpose of this note is to expand the solution $u(x, y)$ into a double Fourier series, to determine the order of magnitude of the Fourier coefficients, and to exhibit a convergent minimizing sequence of the variational problem. Thus, a method is provided for the approximate determination of the solution, and a computable bound for the approximate solutions is exhibited.

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2. Reduction to the problem of solution of an infinite system of linear algebraic equations

Put

$$\vartheta = \pi x/a, \qquad \varphi = \pi y/b.$$

Then (1) and (2) can be written respectively as

$$\left(\frac{\pi}{a}\right)^2 \frac{\partial^2 u}{\partial \vartheta^2} + \left(\frac{\pi}{b}\right)^2 \frac{\partial^2 u}{\partial \varphi^2} - q(\vartheta)\, u = f(\vartheta, \varphi), \tag{3}$$

$$J[\psi] \equiv \iint\limits_G \frac{a\,b}{\pi^2}\left\{\left(\frac{\pi}{a}\,\frac{\partial \psi}{\partial \vartheta}\right)^2 + \left(\frac{\pi}{b}\,\frac{\partial \psi}{\partial \varphi}\right)^2 + q\,\psi^2 + 2 f \psi\right\} d\vartheta\, d\varphi, \tag{4}$$

where, for simplicity, we have used the same letter G to denote the open square, $0<\vartheta<\pi$, $0<\varphi<\pi$, and where its boundary will also be denoted by Γ.

Since it is already known that a unique solution $u(\vartheta, \varphi)$ exists and the partial derivatives of u up to the third order will be Hölder-continuous in G, we may restrict the admissible class of functions. Accordingly, as admissible functions we consider $\psi(\vartheta, \varphi)$ that vanish on Γ and that have the Fourier series expansions [4]

$$\psi(\vartheta, \varphi) = \sum_{m,n=1}^{\infty} x_{m,n} \sin m\vartheta \sin n\varphi$$

in $G+\Gamma$ and

$$\frac{\partial \psi}{\partial \vartheta} = \sum_{m,n=1}^{\infty} m\, x_{m,n} \cos m\vartheta \sin n\varphi,$$

$$\frac{\partial \psi}{\partial \varphi} = \sum_{m,n=1}^{\infty} n\, x_{m,n} \sin m\vartheta \cos n\varphi,$$

in G, where

$$x_{m,n} \equiv \frac{4}{\pi^2} \iint\limits_G \psi \sin m\vartheta \sin n\varphi\, d\vartheta\, d\varphi.$$

From the Parseval relations [4], (4) can be written as

$$\frac{4\pi^2}{a\,b} J = \pi^4 \sum_{m,n=1}^{\infty} \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) x_{m,n}^2 + \\ + 4 \iint\limits_G q \left(\sum_{m,n=1}^{\infty} x_{m,n} \sin m\vartheta \sin n\varphi\right)^2 d\vartheta\, d\varphi - 2\pi^2 \sum_{m,n=1}^{\infty} B_{m,n}\, x_{m,n},$$

where

$$-B_{m,n} \equiv \frac{4}{\pi^2} \iint\limits_G f \sin m\vartheta \sin n\varphi\, d\vartheta\, d\varphi.$$

The vanishing of the variation of J with respect to each parameter $x_{m,n}$ yields [5]

$$\pi^4 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) x_{m,n} + 4 \iint\limits_G (q \sin m\vartheta \sin n\varphi)\, u\, d\vartheta\, d\varphi = \pi^2 B_{m,n}, \tag{5}$$

for $m, n = 1, 2, \ldots$. Again, from the Parseval relations for the functions $q \sin m\vartheta$ sin $n\varphi$ and u, (5) can be written as

$$\pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) x_{m,n} + \sum_{r=1}^{\infty} A_{m,r}\, x_{r,n} = B_{m,n} \qquad (m, n = 1, 2, \ldots), \tag{6}$$

where

$$A_{m,r} \equiv \frac{2}{\pi} \int_0^{\pi} q \sin m\vartheta \sin r\vartheta \, d\vartheta .$$

This shows that the Fourier coefficients of the solution $u(\vartheta, \varphi)$ of our boundary-value problem are determined by the linear algebraic equations in (6).

3. Order of magnitude of the Fourier coefficients $x_{m,n}$

The main result of this section is to show that the unique bounded solution, $\{x_{m,n}\}$, of (6) is such that

$$x_{m,n} = O\left(\frac{1}{m\,n\,(m^2/a^2 + n^2/b^2)}\right). \tag{7}$$

We shall see that this estimate is the best possible if $q(\vartheta)$ or $f(\vartheta, \varphi)$ satisfies only the conditions stated at the beginning of this paper. Furthermore, this result will enable us to obtain a computable bound for the maximum error and to estimate the maximum rate of convergence of the approximate solutions as defined in this paper.

To prove the assertion (7), we shall make use of the theorems of KOCH and HILBERT [*6*] and put

$$\bar{x}_{m\,n} \equiv \pi^2 (m^2/a^2 + n^2/b^2)\, x_{m,n}, \qquad \bar{A}_{m,r,n} \equiv \frac{A_{m,r}}{\pi^2 (r^2/a^2 + n^2/b^2)}. \tag{8}$$

Under these transformations (6) becomes

$$\bar{x}_{m,n} + \sum_{r=1}^{\infty} \bar{A}_{m,r,n}\, \bar{x}_{r,n} = B_{m,n} \qquad (m, n = 1, 2, \ldots). \tag{9}$$

It is true that the existence and uniqueness of the bounded solution of the system (6) is a necessary consequence of the existence and uniqueness proof for the solution of our variational problem. It is also clear that a bounded solution, $\{\bar{x}_{m,n}\}$, of the system (9) will yield a bounded solution, $\{x_{m,n}\}$, of the system (6). However, a bounded solution, $\{x_{m,n}\}$, of the system (6) need not be a bounded solution of the system (9) under the transformation (8). Accordingly, the existence and uniqueness of the bounded solution to (9) has to be proved. The purpose of this proof is to obtain the order of magnitude of the Fourier coefficients $x_{m,n}$ as given by (7). Further, we observe that the index n may be considered as fixed, and the system (9) yields equations for determining the unknowns, $\{\bar{x}_{m,n}\}$, with respect to the index m.

We now show that, for any value of n, the system (9) satisfies Koch's as well as Hilbert's conditions. The justification will be completed if the three series

$$\sum_{m=1}^{\infty} |\bar{A}_{m,m,n}|, \qquad \sum_{m,r=1}^{\infty} (\bar{A}_{m,r,n})^2, \qquad \sum_{m=1}^{\infty} (B_{m,n})^2,$$

are proved to be convergent, and the homogeneous system corresponding to (9) has only a trivial solution. But

$$\sum_{m=1}^{\infty} |\bar{A}_{m,m,n}| = \sum_{m=1}^{\infty} \frac{|A_{m,m}|}{\pi^2 (m^2/a^2 + n^2/b^2)}$$

$$\leqq \left(\int_0^{\pi} q \, d\vartheta \right) \sum_{m=1}^{\infty} \frac{2}{\pi^3 (m^2/a^2 + n^2/b^2)} < \infty;$$

$$\sum_{m,r=1}^{\infty} (\bar{A}_{m,r,n})^2 = \sum_{m,r=1}^{\infty} \left\{ \frac{2 \int_0^{\pi} q \sin m\vartheta \sin r\vartheta \, d\vartheta}{\pi^3 (r^2/a^2 + n^2/b^2)} \right\}^2$$

$$\leqq \sum_{m,r=1}^{\infty} \left\{ \frac{2 \int_0^{\pi} \left(\frac{1}{r} q_{\vartheta} \sin r\vartheta \cos m\vartheta + q \cos r\vartheta \cos m\vartheta \right) d\vartheta}{\pi^3 m r / a^2} \right\}^2$$

$$\leqq \text{const.} \sum_{m,r=1}^{\infty} \frac{1}{m^2 r^2} < \infty;$$

and

$$\sum_{m=1}^{\infty} (B_{m,n})^2 \leqq \sum_{m,n=1}^{\infty} (B_{m,n})^2 = \frac{4}{\pi^2} \iint_G f^2 \, d\vartheta \, d\varphi < \infty.$$

Thus, these three series are indeed convergent for all values of n.

Let $\{\bar{\omega}_{m,n}\}$ be the solution of the homogeneous systems corresponding to (9), *i.e.*,

$$\bar{\omega}_{m,n} + \sum_{r=1}^{\infty} \bar{A}_{m,r,n} \bar{\omega}_{r,n} = 0 \qquad (m, n = 1, 2, \ldots). \tag{10}$$

In order that the left sides of these equations have meaning, the sequences $\{\bar{\omega}_{m,n}\}$ have, of course, to be bounded. In fact, we have stronger conditions to be satisfied by $\{\bar{\omega}_{m,n}\}$, namely,

$$\bar{\omega}_{m,n} = O\left(\frac{1}{n}\right) \qquad (m = 1, 2, \ldots),$$

because

$$\bar{\omega}_{m,n} \leqq \sum_{r=1}^{\infty} |\bar{A}_{m,r,n} \bar{\omega}_{r,n}| = \sum_{r=1}^{\infty} \frac{|A_{m,r} \bar{\omega}_{r,n}|}{\pi^2 (r^2/a^2 + n^2/b^2)}$$

$$\leqq \text{const.} \sum_{r=1}^{\infty} \frac{1}{r^2/a^2 + n^2/b^2} = \frac{\text{const.}}{n} \left\{ \tan^{-1}\left(\frac{b r}{a n} \right) \right\}_{r=1}^{r=\infty}.$$

Define

$$\omega_{m,n} = \frac{\bar{\omega}_{m,n}}{m^2/a^2 + n^2/b^2} \qquad (m, n = 1, 2, \ldots).$$

Then (10) becomes

$$\left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \omega_{m,n} + \sum_{r=1}^{\infty} A_{m,r} \omega_{r,n} / \pi^2 = 0 \qquad (m, n = 1, 2, \ldots). \tag{11}$$

Consider the sequences, $\{m \omega_{m,n}\}$, $\{n \omega_{m,n}\}$, of real numbers. Since

$$m \omega_{m,n} = \frac{m \bar{\omega}_{m,n}}{m^2/a^2 + n^2/b^2} \leqq \text{const.} \frac{m}{n (m^2/a^2 + n^2/b^2)} \leqq \frac{\text{const.}}{m n},$$

$$n \omega_{m,n} = \frac{n \bar{\omega}_{m,n}}{m^2/a^2 + n^2/b^2} \leqq \text{const.} \frac{n}{n (m^2/a^2 + n^2/b^2)} \leqq \frac{\text{const.}}{m n},$$

it follows that

$$\sum_{m,n=1}^{\infty} (m\,\omega_{m,n})^2 < \infty, \qquad \sum_{m,n=1}^{\infty} (n\,\omega_{m,n})^2 < \infty .$$

By the Riesz-Fisher theorem, [7], [4], there are square-integrable functions, $\varrho_\vartheta(\vartheta, \varphi)$, $\varrho_\varphi(\vartheta, \varphi)$, defined in $G+\Gamma$, such that

$$\frac{4}{\pi^2}\iint_G \varrho_\vartheta^2\, d\vartheta\, d\varphi = \sum_{m,n=1}^{\infty} (m\,\omega_{m,n})^2, \qquad \frac{4}{\pi^2}\iint_G \varrho_\varphi^2\, d\vartheta\, d\varphi = \sum_{m,n=1}^{\infty} (n\,\omega_{m,n})^2,$$

and

$$m\,\omega_{m,n} = \frac{4}{\pi^2}\iint_G \varrho_\vartheta \cos m\vartheta \sin n\varphi\, d\vartheta\, d\varphi,$$

$$n\,\omega_{m,n} = \frac{4}{\pi^2}\iint_G \varrho_\varphi \sin m\vartheta \cos n\varphi\, d\vartheta\, d\varphi.$$

Define

$$\varrho(\vartheta, \varphi) = \int_{-\pi}^{\vartheta} \varrho_\vartheta\, d\vartheta;$$

then [4]

$$\varrho(\vartheta, \varphi) = \sum_{m,n=1}^{\infty} \omega_{m,n} \sin m\vartheta \sin n\varphi = \int_{-\pi}^{\varphi} \varrho_\varphi\, d\varphi,$$

$$\omega_{m,n} = \frac{4}{\pi^2}\iint_G \varrho \sin m\vartheta \sin n\varphi\, d\vartheta\, d\varphi.$$

Hence the notations

$$\varrho_\vartheta = \frac{\partial\varrho}{\partial\vartheta}, \qquad \varrho_\varphi = \frac{\partial\varrho}{\partial\varphi}$$

are justified.

By multiplying (11) by $\omega_{m,n}$ and summing over the indices m, n, we obtain

$$\sum_{m,n=1}^{\infty} (m^2/a^2 + n^2/b^2)\,\omega_{m,n}^2 + \sum_{m,n=1}^{\infty} \omega_{m,n}\Bigl(\sum_{r=1}^{\infty} A_{m,r}\,\omega_{r,n}/\pi^2\Bigr) = 0. \tag{12}$$

Define

$$A_{m,n,r,s} = \frac{4}{\pi^2}\iint_G (q \sin m\vartheta \sin n\varphi) \sin r\vartheta \sin s\varphi\, d\vartheta\, d\varphi.$$

Then

$$A_{m,n,r,s} = A_{m,r}\,\delta_{ns},$$

and

$$\frac{1}{\pi^2}\sum_{r=1}^{\infty} A_{m,r}\,\omega_{r,n} = \frac{1}{\pi^2}\sum_{r,s=1}^{\infty} A_{m,n,r,s}\,\omega_{r,s} = \frac{4}{\pi^4}\iint_G (q \sin m\vartheta \sin n\varphi)\,\varrho\, d\vartheta\, d\varphi.$$

This is legitimate because the single and the double series are both absolutely convergent, and they are equal to the integral on the right in view of the Parseval relations for the functions ϱ and $q \sin m\vartheta \sin n\varphi$. Now (12) becomes

$$\frac{4}{\pi^2}\iint_G \{(\varrho_\vartheta/a)^2 + (\varrho_\varphi/b)^2\}\, d\vartheta\, d\varphi + \sum_{m,n=1}^{\infty} \omega_{m,n}\,\frac{4}{\pi^4}\iint_G q\,\varrho \sin m\vartheta \sin n\varphi\, d\vartheta\, d\varphi = 0.$$

Applying Parseval relations for the functions ϱ and $q\varrho$ to this equation, we have

$$\frac{4}{\pi^2}\iint\limits_G \left\{\left(\frac{\pi}{a}\,\varrho_\vartheta\right)^2+\left(\frac{\pi}{b}\,\varrho_\varphi\right)^2+q\,\varrho^2\right\}d\vartheta\,d\varphi=0.$$

Since $q\geqq 0$, this is possible if and only if

$$\iint\limits_G \left(\frac{\partial\varrho}{\partial\vartheta}\right)^2 d\vartheta\,d\varphi=\iint\limits_G \left(\frac{\partial\varrho}{\partial\varphi}\right)^2 d\vartheta\,d\varphi=\iint\limits_G q\,\varrho^2\,d\vartheta\,d\varphi=0.$$

Thus

$$\omega_{m,n}=0 \qquad (m,n=1,2,\ldots),$$

and

$$\overline{\omega}_{m,n}=(m^2/a^2+n^2/b^2)\,\omega_{m,n}=0$$

for all finite values of m and n. These results together with the equations in (10) imply

$$\lim_{m\to\infty}\overline{\omega}_{m,n}=\lim_{n\to\infty}\overline{\omega}_{m,n}=\lim_{m,n\to\infty}\overline{\omega}_{m,n}=0.$$

Hence, we have proved that

$$\overline{\omega}_{m,n}=0 \qquad (m,n=1,2,\ldots),$$

and the homogeneous systems corresponding to (9) have only trivial solutions for any value of n.

Now, Hilbert's theorem asserts that for any fixed value of n, the system (9) has only one solution, $\{\overline{x}_{m,n}\}$, such that $\sum\limits_{m=1}^{\infty}\overline{x}_{m,n}^2<\infty$.

To prove the assertion given by (7), we consider the inequality

$$(13)\qquad |\overline{x}_{m,n}|\leqq|B_{m,n}|+\left|\sum_{r=1}^{\infty}\overline{A}_{m,r,n}\,\overline{x}_{r,n}\right| \qquad (m,n=1,2,\ldots).$$

By the Lagrange-Cauchy inequality [6], we have

$$\left|\sum_{r=1}^{\infty}\overline{A}_{m,r,n}\,\overline{x}_{r,n}\right|^2\leqq\sum_{r=1}^{\infty}(\overline{A}_{m,r,n})^2\sum_{r=1}^{\infty}(\overline{x}_{r,n})^2.$$

But, for all values of n,

$$\sum_{r=1}^{\infty}(\overline{A}_{m,r,n})^2\leqq\text{const.}\sum_{r=1}^{\infty}\frac{1}{m^2r^2}=O\left(\frac{1}{m^2}\right),\qquad \sum_{r=1}^{\infty}(\overline{x}_{r,n})^2<\infty;$$

thus

$$(14)\qquad \left|\sum_{r=1}^{\infty}\overline{A}_{m,r,n}\,\overline{x}_{r,n}\right|=O\left(\frac{1}{m}\right) \qquad (n=1,2,\ldots).$$

Further it can be verified by partial integration that

$$(15)\qquad B_{m,n}=O\left(\frac{1}{m\,n}\right).$$

Now from (13), (14) and (15), we have

$$\overline{x}_{m,n}=O\left(\frac{1}{m}\right) \qquad (n=1,2,\ldots).$$

Using this result, we can write

$$
\begin{aligned}
\left|\sum_{r=1}^{\infty} \bar{A}_{m,r,n}\, \bar{x}_{r,n}\right| &\leqq \sum_{r=1}^{\infty} \left| \frac{2\,\bar{x}_{r,n} \int\limits_0^{\pi} (q_\vartheta \sin r\,\vartheta \cos m\,\vartheta + q\, r \cos r\,\vartheta \cos m\,\vartheta)\, d\vartheta}{\pi^3\, m\,(r^2/a^2+n^2/b^2)} \right| \\
&\leqq \text{const.} \sum_{r=1}^{\infty} \left| \frac{\int\limits_0^{\pi} \left(\frac{1}{r}\, q_\vartheta \sin r\,\vartheta \cos m\,\vartheta + q \cos r\,\vartheta \cos m\,\vartheta\right) d\vartheta}{m\, n^2\,(1+b^2 r^2/a^2 n^2)} \right| \qquad (16) \\
&\leqq \frac{\text{const.}}{m\, n} \left\{\tan^{-1}\left(\frac{b\, r}{a\, n}\right)\right\}_{r=1}^{r=\infty} = O\left(\frac{1}{m\, n}\right).
\end{aligned}
$$

Now (7) is a direct consequence of (13), (15), (16) and (18). As can be seen from the equations in (9), the estimate (7) is the best possible if $f(\vartheta, \varphi)$ or $q(\vartheta)$ satisfies only the conditions imposed on them.

Because of the absolute convergence of the corresponding majorant series,

$$
\sum_{m,n=1}^{\infty} \frac{\text{const.}}{m\, n\,(m^2/a^2+n^2/b^2)}\,, \qquad \sum_{m,n=1}^{\infty} \frac{\text{const.}}{m\,(m^2/a^2+n^2/b^2)}\,, \qquad \sum_{m,n=1}^{\infty} \frac{\text{const.}}{n\,(m^2/a^2+n^2/b^2)}\,,
$$

the series

$$
\sum_{m,n=1}^{\infty} x_{m,n} \sin m\,\vartheta \sin n\,\varphi\,, \qquad \sum_{m,n=1}^{\infty} m\, x_{m,n} \cos m\,\vartheta \sin n\,\varphi\,, \qquad \sum_{m,n=1}^{\infty} n\, x_{m,n} \sin m\,\vartheta \cos n\,\varphi
$$

converge absolutely and uniformly in G. This also shows some resemblance between our solution and the solution of the equation $\Delta u = f(x, y)$, [*11*].

4. Method for finding an approximate solution-minimizing sequence

Even though the convergent behavior of the left sides in (9) is known, in general such an infinite system can not be solved exactly. For approximate solutions, we consider the sequence, $\{\psi_{(N)}\}$, defined by

$$
\psi_{(N)} \equiv \sum_{m,n=1}^{N} x_{m,n}^{(N)} \sin m\,\vartheta \sin n\,\varphi\,, \tag{17}
$$

where the coefficients, $x_{m,n}^{(N)}$, are determined by the finite system of linear algebraic equations

$$
\pi^2 (m^2/a^2 + n^2/b^2)\, x_{m,n}^{(N)} + \sum_{r=1}^{N} A_{m,r}\, x_{r,n}^{(N)} = B_{m,n}\,, \tag{18}
$$

$m, n = 1, 2, \ldots, N$. By using the same technique as before, it can be shown that the finite system (18) always has a unique solution for any value of the integer N. Moreover, using KOCH's results [*6*] on normal determinants, we can prove that

$$
x_{m,n}^{(N)} \to x_{m,n} \quad \text{as} \quad N \to \infty\,.
$$

In fact, it is easy to see that $\{\psi_{(N)}\}$ is a minimizing sequence for the Dirichlet integral (4). Accordingly [*1*], $\psi_{(N)}$ is an approximate solution in the sense that $\|\psi_{(N)} - u\|$ approaches zero as $N \to \infty$. But we shall obtain stronger results than these in the next section.

The method of finding an approximate solution by solving the truncated finite system (18) is known [*8*] to be convenient. The essential feature of the approximate solution $\psi_{(N)}$ is that it has the closed form (17). Also we note that the same method for finding approximate solutions can be applied to the non-homogeneous boundary conditions, provided the boundary-value function, g, defined on Γ, can be extended to a function defined on $G+\Gamma$ in such a way that its partial derivatives up to the third order are Hölder-continuous in $G+\Gamma$.

5. Maximum-error estimation for the approximate solutions

KANTOROVICH proved a general theorem [*9*], [*10*] about the rate of convergence of the approximate solutions found by minimizing the Dirichlet integral. However, for this problem a stronger result can be obtained. Inspired by KRYLOV'S work [*9*] on ordinary differential equations, we shall present a computable bound for the maximum error, $\max|\psi_{(N)}-u|$. To show that this bound will also indicate the maximum rate of the convergence of the approximate solutions, $\psi_{(N)}$, defined by (17), we shall compare $\psi_{(N)}$ with the segment of the double sine series of the exact solution u.

The systems of equations in (6) and (18) can be written as

$$x_{m,n}+\frac{1}{\pi^2(m^2/a^2+n^2/b^2)}\sum_{r=1}^{\infty}A_{m,r}\,x_{r,n}=B_{m,n}/\pi^2(m^2/a^2+n^2/b^2), \tag{19}$$

$m,n=1,2,\ldots,$ and

$$x^{(N)}_{m,n}+\frac{1}{\pi^2(m^2/a^2+n^2/b^2)}\sum_{r=1}^{\infty}A_{m,r}\,x^{(N)}_{r,n}=B_{m,n}/\pi^2(m^2/a^2+n^2/b^2) \tag{20}$$

$(m,n=1,2,\ldots,N)$, where $x^{(N)}_{r,n}\equiv 0$ for $r\geqq N+1$ or $n\geqq N+1$. Denote by $r_{(N)}$ and $R_{(N)}$ the errors of the approximate solutions and the errors of the segment of the double-sine series of u, i.e.,

$$r_{(N)}\equiv\sum_{m,n=1}^{\infty}(x^{(N)}_{m,n}-x_{m,n})\sin m\vartheta\sin n\varphi,$$

$$R_{(N)}\equiv-\sum_{m,n=N+1}^{\infty}x_{m,n}\sin m\vartheta\sin n\varphi.$$

It is not difficult to see that we can construct the following two identities from the systems of equations in (19) and (20), namely,

$$\begin{aligned}r_{(N)}&+\sum_{m,n=1}^{N}\frac{4\sin m\vartheta\sin n\varphi}{\pi^4(m^2/a^2+n^2/b^2)}\iint_G q\,r_{(N)}\sin m\xi\sin n\eta\,d\xi\,d\eta\\&-\sum_{m,n=N+1}^{\infty}\frac{4\sin m\vartheta\sin n\varphi}{\pi^4(m^2/a^2+n^2/b^2)}\iint_G q\,u\sin m\xi\sin n\eta\,d\xi\,d\eta\\&=-\sum_{m,n=N+1}^{\infty}\frac{B_{m,n}\sin m\vartheta\sin n\varphi}{\pi^2(m^2/a^2+n^2/b^2)};\end{aligned} \tag{21}$$

$$\begin{aligned}R_{(N)}&-\sum_{m,n=N+1}^{\infty}\frac{4\sin m\vartheta\sin n\varphi}{\pi^4(m^2/a^2+n^2/b^2)}\iint_G q\,u\sin m\xi\sin n\eta\,d\xi\,d\eta\\&=-\sum_{m,n=N+1}^{\infty}\frac{B_{m,n}\sin m\vartheta\sin n\varphi}{\pi^2(m^2/a^2+n^2/b^2)}.\end{aligned} \tag{22}$$

From the two identities (21) and (22), we have [*4*], [*7*]

$$
\begin{aligned}
|r_{(N)}| &\leqq |R_{(N)}| + \left| \sum_{m,n=1}^{N} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \iint_G q\, r_{(N)} \sin m\xi \sin n\eta \, d\xi \, d\eta \right| \\
&\leqq |R_{(N)}| + \left| \iint_G \left[\sum_{m,n=1}^{N} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \sin m\xi \sin n\eta \right] q\, r_{(N)} \, d\xi \, d\eta \right| \\
&\leqq |R_{(N)}| + \left[\iint_G \left(\sum_{m,n=1}^{N} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \sin m\xi \sin n\eta \right)^2 d\xi \, d\eta \right]^{\frac{1}{2}} \times \\
&\qquad \times \left[\iint_G (q\, r_{(N)})^2 \, d\xi \, d\eta \right]^{\frac{1}{2}} \\
&\leqq |R_{(N)}| + \left[\sum_{m,n=1}^{N} \frac{4 (\sin m\vartheta \sin n\varphi)^2}{\pi^6 (m^2/a^2 + n^2/b^2)^2} \right]^{\frac{1}{2}} \max q^{\frac{1}{2}} \left[\iint_G q\, r_{(N)}^2 \, d\xi \, d\eta \right]^{\frac{1}{2}} \\
&\leqq |R_{(N)}| + \frac{a\,b}{\pi^3} \max q^{\frac{1}{2}} \left[\iint_G q\, r_{(N)}^2 \, d\xi \, d\eta \right]^{\frac{1}{2}} .
\end{aligned} \tag{23}
$$

Thus, to estimate $|r_{(N)}|$ we need only to give estimates for the first and second terms on the right-hand side of the last inequality in (23).

First, we give an estimate for $\iint_G q r_{(N)}^2 \, d\vartheta \, d\varphi$. Multiplying both sides of the identity (21) by $q r_{(N)}$ and integrating them over the domain G, we obtain

$$
\begin{aligned}
&\left| \iint_G q\, r_{(N)}^2 \, d\vartheta \, d\varphi + \sum_{m,n=1}^{N} \frac{4 \left(\iint_G q\, r_{(N)} \sin m\vartheta \sin n\varphi \, d\vartheta \, d\varphi \right)^2}{\pi^4 (m^2/a^2 + n^2/b^2)} \right| \\
&\quad \leqq \left| \iint_G q\, r_{(N)} \left(\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \iint_G q\, u \sin m\xi \sin n\eta \, d\xi \, d\eta \right) d\vartheta \, d\varphi \right| + \\
&\qquad + \left| \iint_G q\, r_{(N)} \left(\sum_{m,n=N+1}^{\infty} \frac{B_{m,n} \sin m\vartheta \sin n\varphi}{\pi^2 (m^2/a^2 + n^2/b^2)} \right) d\vartheta \, d\varphi \right| .
\end{aligned}
$$

Since the first and the second terms on the left side of this inequality are both positive, we have

$$
\begin{aligned}
&\iint_G q\, r_{(N)}^2 \, d\vartheta \, d\varphi \\
&\quad \leqq \left| \iint_G q\, r_{(N)} \left(\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \iint_G q\, u \sin m\xi \sin n\eta \, d\xi \, d\eta \right) d\vartheta \, d\varphi \right| + \\
&\qquad + \left| \iint_G q\, r_{(N)} \left(\sum_{m,n=N+1}^{\infty} \frac{B_{m,n} \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \right) d\vartheta \, d\varphi \right| .
\end{aligned} \tag{24}
$$

Now we give estimates for the first and the second terms on the right-hand side of the inequality (24). The first term is less than or equal to

$$\left[\iint\limits_G (q\, r_{(N)})^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \times$$

$$\times \left[\iint\limits_G \left(\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2 + n^2/b^2)} \iint\limits_G q\, u \sin m\xi \sin n\eta\, d\xi\, d\eta\right)^2 d\vartheta\, d\varphi\right]^{\frac{1}{2}}$$

$$\text{(25)}\qquad \leqq \max q^{\frac{1}{2}} \left[\iint\limits_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \left[\sum_{m,n=N+1}^{\infty} \left(\frac{2 \iint_G q\, u \sin m\xi \sin n\eta\, d\xi\, d\eta}{\pi^3 (m^2/a^2 + n^2/b^2)}\right)^2\right]^{\frac{1}{2}}$$

$$\leqq \max q^{\frac{1}{2}} \left[\iint\limits_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \left[\sum_{m,n=N+1}^{\infty} \left(\frac{2 \iint_G q\, u_\varphi \sin m\vartheta \cos n\varphi\, d\vartheta\, d\varphi}{\pi^3 n (m^2/a^2 + n^2/b^2)}\right)^2\right]^{\frac{1}{2}}$$

$$\leqq \max q^{\frac{1}{2}} \left[\iint\limits_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \frac{a\, b \|q\|\, \|u_\varphi\|}{\pi^3 \sqrt{3}\, (N+1)^2},$$

where we have made use of the Riesz-Fisher theorem and where $\|q\|$ denotes the L_2-norm of q. To estimate $\|u_\varphi\|$, we consider the Dirichlet integral (4). If u minimizes this integral, then [9]

$$\iint\limits_G \left\{\left(\frac{\pi}{a}\right)^2 u_\vartheta^2 + \left(\frac{\pi}{b}\right)^2 u_\varphi^2 + q\, u^2 + f\, u\right\} d\vartheta\, d\varphi = 0.$$

Consequently,

$$\left(\frac{\pi}{b}\right)^2 \|u_\varphi\|^2 \leqq \left|\iint\limits_G f\, u\, d\vartheta\, d\varphi\right| \leqq \|f\|\, \|u\|.$$

But

$$\frac{4}{\pi^2} \|u\|^2 = \sum_{u,n=1}^{\infty} (x_{mn})^2 \leqq \sum_{m,n=1}^{\infty} n^2 (x_{mn})^2 = \frac{4}{\pi^2} \|u\|^2.$$

Hence

$$\|u_\varphi\| \leqq \left(\frac{b}{\pi}\right)^2 \|f\|.$$

From this inequality and the last inequality in (25), we know that the first term on the right-hand side of the inequality (24) is less than or equal to

$$\text{(26)}\qquad \max q^{\frac{1}{2}} \left[\iint\limits_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \frac{a\, b^3 \|q\|\, \|f\|}{\pi^5 \sqrt{3}\, (N+1)^2}.$$

The second term on the right-hand side of the inequality (24) is less than or equal to

$$\left[\iint\limits_G (q\, r_{(N)})^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \left[\iint\limits_G \left(\sum_{m,n=N+1}^{\infty} \frac{B_{m,n} \sin m\vartheta \sin n\varphi}{\pi^2 (m^2/a^2 + n^2/b^2)}\right)^2 d\vartheta\, d\varphi\right]$$

$$\leqq \max q^{\frac{1}{2}} \left[\iint\limits_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \left[\sum_{m,n=N+1}^{\infty} \frac{(B_{m,n})^2/4}{\pi^2 (m^2/a^2 + n^2/b^2)^2}\right]^{\frac{1}{2}}.$$

Since

$$(B_{m,n})^2 = \left(\frac{4}{\pi^2} \iint\limits_G f \sin m\vartheta \sin n\varphi\, d\vartheta\, d\varphi\right)^2$$

$$\leqq \left(\frac{4}{\pi^2}\right)^2 [\|f(0,\varphi)\| + \|f(\pi,\varphi)\| + \|f_\vartheta\|]^2/m^2,$$

the second term on the right-hand side of (24) is less than or equal to

$$
\begin{aligned}
&\max q^{\frac{1}{2}} \left[\iint_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \frac{2}{\pi^3} \left[\sum_{m,n=N+1}^{\infty} \frac{(\|f(0,\varphi)\|+\|f(\pi,\varphi)\|+\|f_\vartheta\|)^2}{m^2(m^2/a^2+n^2/b^2)^2}\right]^{\frac{1}{2}} \\
&\qquad \leqq \frac{a\, b\,(\|f(0,\varphi)\|+\|f(\pi,\varphi)\|+\|f_\vartheta\|)}{\sqrt{3}\,\pi^3\,(N+1)^2} \leqq \max q^{\frac{1}{2}} \left[\iint_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}}.
\end{aligned}
\tag{27}
$$

Finally, from the inequalities (24), (26) and (27), we see that

$$
\left[\iint_G q\, r_{(N)}^2\, d\vartheta\, d\varphi\right]^{\frac{1}{2}} \leqq \frac{\max q^{\frac{1}{2}}}{(N+1)^2} \left\{\frac{a\, b^3 \|q\|\,\|f\|}{\sqrt{3}\,\pi^5} + \frac{a\, b\,(\|f(0,\varphi)\|+\|f(\pi,\varphi)\|+\|f_\vartheta\|)}{\sqrt{3}\,\pi^3}\right\},
\tag{28}
$$

and the estimate for the second term on the right-hand side of the last inequality in (23) is completed.

From the inequalities in (23) and (28), we see that to estimate $|r_{(N)}|$ we need only to give an estimate for $|R_{(N)}|$. From (22), we have

$$
\begin{aligned}
|R_{(N)}| \leqq &\left|\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2+n^2/b^2)} \iint_G q\, u \sin m\xi \sin n\eta\, d\xi\, d\eta\right| + \\
&+ \left|\sum_{m,n=N+1}^{\infty} \frac{B_{m,n} \sin m\vartheta \sin n\varphi}{\pi^2 (m^2/a^2+n^2/b^2)}\right|.
\end{aligned}
\tag{29}
$$

Now, the first term on the right-hand side of (29) is equal to

$$
\begin{aligned}
&\left|\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4\, n\, (m^2/a^2+n^2/b^2)} \iint_G q\, u_\eta \sin m\xi \cos n\eta\, d\xi\, d\eta\right| \\
&\leqq \left[\iint_G \left(\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4\, n\, (m^2/a^2+n^2/b^2)} \sin m\xi \cos n\eta\right)^2 d\xi\, d\eta\right]^{\frac{1}{2}} \left[\iint_G (q\, u_\eta)^2\, d\xi\, d\eta\right]^{\frac{1}{2}} \\
&\leqq \left[\sum_{m,n=N+1}^{\infty} \frac{4 (\sin m\vartheta \sin n\varphi)^2}{\pi^6\, n^2\, (m^2/a^2+n^2/b^2)^2}\right|^{\frac{1}{2}} \|q\|\, \|u_\varphi\| \\
&\leqq \frac{a\, b^3 \|q\|\, \|f\|}{\pi^5 \sqrt{3}\,(N+1)^2}\,;
\end{aligned}
\tag{30}
$$

and the second term on the right-hand side of (29) is equal to

$$
\begin{aligned}
&\left|\sum_{m,n=N+1}^{\infty} \frac{4 \sin m\vartheta \sin n\varphi}{\pi^4 (m^2/a^2+n^2/b^2)} \iint_G f \sin m\xi \sin n\eta\, d\xi\, d\eta\right| \\
&\quad\leqq \sum_{m,n=N+1}^{\infty} \frac{4\,|f(i,j)|}{\pi^4\, m\, n\, (m^2/a^2+n^2/b^2)} + \sum_{m,n=N+1}^{\infty} \frac{4\,(\|f_\varphi(0,\varphi)\|+\|f_\varphi(\pi,\varphi)\|)}{\pi^4\, m\, n\, (m^2/a^2+n^2/b^2)} + \\
&\qquad + \left[\sum_{m,n=N+1}^{\infty} \frac{4 (\sin m\vartheta \sin n\varphi)^2}{\pi^6\, m^2\, (m^2/a^2+n^2/b^2)^2}\right]^{\frac{1}{2}} \|f_\vartheta\| \\
&\quad\leqq \frac{2ab\,|f(i,j)|}{\pi^4 (N+1)^2} + \frac{2ab\,(\|f_\varphi(0,\varphi)\|+\|f_\varphi(\pi,\varphi)\|)}{\pi^4 (N+1)^2} + \frac{a\, b\, \|f_\vartheta\|}{\pi^3 \sqrt{3}\,(N+1)^2} \\
&\quad\leqq \frac{2ab}{\pi^4 (N+1)^2} \left\{|f(i,j)| + \|f_\varphi(0,\varphi)\| + \|f_\varphi(\pi,\varphi)\| + \frac{\pi \|f_\vartheta\|}{2\sqrt{3}}\right\},
\end{aligned}
\tag{31}
$$

where

$$|f(i,j)| \equiv |f(0,0)| + |f(0,\pi)| + |f(\pi,0)| + |f(\pi,\pi)|,$$

$$\|f_\varphi(0,\varphi)\|^2 \equiv \int_0^\pi \left(\frac{\partial f}{\partial \varphi}\right)^2_{\vartheta=0} d\varphi, \qquad \|f_\varphi(\pi,\varphi)\|^2 \equiv \int_0^\pi \left(\frac{\partial f}{\partial \varphi}\right)^2_{\vartheta=\pi} d\varphi.$$

Hence

$$(32) \qquad |R_{(N)}| \leqq \frac{2ab}{\pi^4 (N+1)^2} \left\{ \frac{b^2 \|q\| \, \|f\|}{2\sqrt{3}\,\pi} + |f(i,j)| + \|f_\varphi(0,\varphi)\| + \|f_\varphi(\pi,\varphi)\| + \frac{\pi \|f_\vartheta\|}{2\sqrt{3}} \right\},$$

and the estimates for $|R_{(N)}|$ are completed.

Finally, from the inequalities (23), (32) and (28), we see that

$$(33) \qquad \begin{aligned} |r_{(N)}| \leqq \frac{2ab}{\pi^4 (N+1)^2} \Big\{ &\Big(\frac{b^2}{2\sqrt{3}\,\pi} + \frac{a b^3}{\sqrt{3}\,\pi^4} \max q \Big) \|q\| \, \|f\| + |f(i,j)| + \|f_\varphi(0,\varphi)\| + \\ &+ \|f_\varphi(\pi,\varphi)\| + \frac{\pi \|f_\vartheta\|}{2\sqrt{3}} + \frac{ab \max q}{\sqrt{3}\,\pi^2} \left(\|f(0,\varphi)\| + \|f(\pi,\varphi)\| + \|f_\vartheta\| \right) \Big\}. \end{aligned}$$

This is the final form of our estimation. Since q and f are known functions, the right-hand side of this inequality gives a computable bound for the maximum error, $\max |r_{(N)}|$, and the integer N indicates the rate of convergence of the approximate solutions $\psi_{(N)}$. If $f = f(\vartheta)$ only, as in the case of REYNOLDS' equation for finite partial journal bearings, then

$$(34) \qquad \begin{aligned} |r_{(N)}| \leqq \frac{2ab}{\pi^4 (N+1)^2} \Big\{ &\Big(\frac{b^2}{2\sqrt{3}\,\pi} + \frac{a b^3}{\sqrt{3}\,\pi^4} \max q \Big) \|q\| \, \|f\| + 3|f(0)| + \\ &+ 3|f(\pi)| + \frac{\pi \|f_\vartheta\|}{2\sqrt{3}} + \frac{ab \max q}{\sqrt{3}\,\pi^2} \|f_\vartheta\| \Big\}. \end{aligned}$$

To show that the inequalities (33) or (34) also indicate the maximum rate of the convergence of $\psi_{(N)}$ to the exact solution u, we compare the approximate solutions $\psi_{(N)}$ with the segments of the double-sine series of u. From the inequality (23), we have

$$\max |r_{(N)}| \leqq \max |R_{(N)}| + \frac{2ab}{\pi^3} \max q^{\frac{1}{2}} \left[\iint_G q \, r^2_{(N)} \, d\vartheta \, d\varphi \right]^{\frac{1}{2}}.$$

Also we have

$$\iint_G q \, r^2_{(N)} \, d\vartheta \, d\varphi + \sum_{m,n=1}^{N} \frac{4 \left(\iint_G q \, r_{(N)} \sin m\vartheta \sin n\varphi \, d\vartheta \, d\varphi \right)^2}{\pi^4 (m^2/a^2 + n^2/b^2)} = \iint_G q \, r_{(N)} R_{(N)} \, d\vartheta \, d\varphi.$$

Hence

$$\iint_G q \, r^2_{(N)} \, d\vartheta \, d\varphi \leqq \|R_{(N)}\|^2 \max q.$$

Thus

$$\max |r_{(N)}| \leqq \max |R_{(N)}| + \frac{2ab}{\pi^3} \|R_{(N)}\| \max q.$$

Consequently,

$$(35) \qquad \begin{aligned} \frac{\max |r_{(N)}|}{\max |R_{(N)}|} &\leqq 1 + \frac{2ab \|R_{(N)}\|}{\pi^3 \max |R_{(N)}|} \max q \\ &\leqq 1 + \frac{2ab}{\pi} \max q. \end{aligned}$$

Similarly,

$$|R_{(N)}| \leqq |r_{(N)}| + \left| \sum_{m,n=1}^{\infty} \frac{4\sin m\vartheta \sin n\varphi}{\pi^4(m^2/a^2+n^2/b^2)} \iint_G q\, r_{(N)} \sin m\xi \sin n\eta \, d\xi \, d\eta \right|$$

$$\leqq \max|r_{(N)}| + \frac{a\,b}{\pi^3} \|q\| \, \|r_{(N)}\|.$$

Hence

$$\frac{\max|R_{(N)}|}{\max|r_{(N)}|} \leqq 1 + \frac{a\,b}{\pi^3} \|q\| \frac{\|r_{(N)}\|}{\max|r_{(N)}|} \leqq 1 + \frac{a\,b\,\|q\|}{\pi}. \tag{36}$$

From the inequalities (35) and (36), we see that $r_{(N)}$ converges to zero at the same rate as $R_{(N)}$. That is, the rate of convergence of the approximate solutions $\psi_{(N)}$ to the exact solution u is the same as that of the double Fourier series for u. Since the order of magnitude of the Fourier coefficients of u is given by (7), we can conclude that the estimates given by (33) and (34) are the best possible insofar as the rate of convergence is concerned.

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